UNIVERSIDAD AUTÓNOMA DE MADRID

Facultad de Ciencias
Departamento de Matemáticas

Coefficient Problems in Geometric Function Theory

Doctoral Thesis of

Iason Efraimidis

Supervised by Professor

Dragan Vukotić Jovšić

Madrid

June 2017

Contents

Introduction

This thesis treats a number of extremal problems in Geometric Function Theory, mainly concerning univalent functions but, also, other analytic or harmonic mappings in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Significant for the development of the theory has been the Bieberbach conjecture which states that the coefficients of normalized univalent functions must satisfy $|a_n| \leq n$. The conjecture was stated in 1916 and proved by de Branges in 1984, leaving behind an arsenal of techniques, such as the Loewner theory and a plethora of new variational methods, among others. Our aim will be to study two conjectures related to Bieberbach's, namely the conjectures of Zalcman and of Bombieri, in Chapters 2 and 3, respectively. Also related to univalent functions is the Carathéodory class of functions with positive real part. In Chapter 1 we will present some results on the coefficients of functions in this class.

Since the 1980's researches in univalent functions tried to extend this theory to harmonic mappings, especially after an influential paper of Clunie and Sheil-Small [18]. In Chapter 4 we will discuss a new definition of harmonic Bloch-type mappings and some related results.

Livingston's inequalities. In the first chapter we are concerned with two inequalities proved by Livingston [55, 56] for functions in the class $\mathcal{P}$, that is, analytic functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots, \qquad z \in \mathbb{D},$$

which satisfy $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. A well-known theorem of Carathéodory states that $|p_n| \leq 2$ for all $n \geq 1$, while the first of Livingston's results states that $|p_n - p_k p_{n-k}| \leq 2$ for all $0 \leq k \leq n-1$. We will see that this can be generalized to

$$|p_n - wp_kp_{n-k}| \leq 2\max\{1, |1-2w|\}, \quad w \in \mathbb{C}. \tag{1}$$

We will also see a characterization of the case of equality for every $w \in \mathbb{C}$, which appears to be new even in the case when $w = 1$. A simple lemma can be used to prove that infinitely many conditions in (2) are equivalent to the single condition

$$\left|p_n - \frac{p_kp_{n-k}}{2}\right| \leq 2 - \frac{|p_kp_{n-k}|}{2}.$$

Moreover, Livingston's second result provides the sharp bound for some functionals related to determinants of Hessenberg (almost-triangular) matrices with entries the coefficients p_n. Both of Livingston's results will be proved in a substantially simpler fashion. Inequality (2) also provides a new and simpler proof of a result of Brown [12], which states that

$$|e^{i\nu}p_{n+m} - p_n| \leq 2\sqrt{2 - \mathrm{Re}\,(e^{i\nu}p_m)}$$

for all $n, m \geq 1, \nu \in \mathbb{R}$. We will provide some applications to self-maps of the unit disk, although the main applications -and source of motivation- will be presented in Chapter 2, which is related to Zalcman's conjecture.

Zalcman's conjecture. For normalized univalent analytic functions

$$f(z) = z + a_2z^2 + a_3z^3 \ldots, \qquad z \in \mathbb{D},$$

that is, functions in the well-known class S, the Zalcman conjecture states that the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2$$

should be true for all $n \geq 2$. Its importance lies in the fact that, if assumed true, it implies Bieberbach's conjecture (see [13]). It is known to be true for small values of n and various subclasses of S (see [32] and the references therein). Brown and Tsao [13] and Ma [58] proposed the study of the generalized Zalcman functional $\Phi(f) = \lambda a_na_m - a_{n+m-1}$ for $\lambda > 0$. In Chapter 2 we continue this study and, using the results of Chapter 1, extend it to complex values of the parameter λ. We will obtain sharp estimates for the closed convex hulls of the classes of starlike and convex functions of a given order -two classes that also contain non-univalent functions- as well as for the Hurwitz and Noshiro-Warschawksi classes. For example, applying inequality (2) we will obtain the following result: If f is in the closed convex hull of the class of starlike functions then for all $m, n \geq 2$ we have that

$$|\lambda a_m a_n - a_{m+n-1}| \leq (m+n-1) \max \left\{ 1, \left| 1 - \frac{mn}{m+n-1} \lambda \right| \right\} \quad \text{for all } \lambda \in \mathbb{C}\,.$$

This answers a question of Ma [58] on the smallest positive λ for which one of his estimates holds. The merit in considering complex parameters λ, apart from the sake of generalization, is that this infinite number of inequalities is equivalent to the single inequality

$$\left| \frac{a_m a_n}{mn} - \frac{a_{m+n-1}}{m+n-1} \right| + \frac{|a_m a_n|}{mn} \leq 1\,,$$

which has some interest of its own. In the same fashion, each of our results that is formulated as an inequality that holds for all $\lambda \in \mathbb{C}$ will be enunciated in an equivalent way as a single new inequality. We will observe that our theorem for the Hurwitz class reflects a new phenomenon for the generalized functional $\lambda a_m a_n - a_{m+n-1}$: the sharp bounds obtained differ in an essential way in the case $m \neq n$ from the case $m = n$.

We will show that the generalized Zalcman conjecture is asymptotically true for every complex value of λ and is also equivalent to other related statements which may provide further insight into the problem. We will improve upon the observation that the Zalcman conjecture implies the Bieberbach conjecture by showing that this implication passes through three related but weaker conjectures than Zalcman's.

Bombieri's conjecture. Long before the final solution of the Bieberbach conjecture by de Branges, many researchers concentrated their efforts in solving a weaker problem: the Koebe function

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$$

should be, at least, a *local* maximum in S for the functional $\operatorname{Re} a_n$. Powerful methods were employed by Duren, Garabedian, Ross and Schiffer, among others, but the problem finally yielded to Bombieri [8] in 1967, who combined the Loewner theory with the variational techniques of Duren and Schiffer [26].

In the same article [8], Bombieri conjectured in very precise terms what the behavior of the coefficients of univalent functions should be close to the Koebe function. Namely, he proposed that the two real numbers

$$\sigma_{mn} := \liminf_{f\to K} \frac{n - \operatorname{Re} a_n}{m - \operatorname{Re} a_m} \qquad \text{and} \qquad B_{mn} := \min_{t\in\mathbb{R}} \frac{n\sin t - \sin(nt)}{m\sin t - \sin(mt)}$$

should be equal for all $m, n \geq 2$. Although it has been proved that $0 \leq \sigma_{mn} \leq B_{mn}$ and that $\sigma_{mn} = B_{mn}$ for functions with real coefficients (see [64]), the conjecture was disproved by Greiner and Roth [40] in the case $(m, n) = (3, 2)$, while disproofs for the points $(2, 4), (3, 4)$ and $(4, 2)$ were then furnished by Prokhorov and Vasil'ev [65].

Recently, Leung [51] used Bombieri's second variation formula to disprove the conjecture for $n = 2$ and for all $m \geq 3$ and, also, for $n = 3$ and for all odd $m \geq 5$. Complementing his work we will disprove Bombieri's conjecture for all $m > n \geq 2$ which are simultaneously odd or even and, also, for the case when m is odd, n is even and $n \leq \frac{m+1}{2}$. We will mostly make use of trigonometry, but also employ Dieudonné's criterion for the univalence of polynomials.

Harmonic Bloch-type mappings. Let f be a complex-valued harmonic mapping defined in the unit disk $\mathbb{D}$, meaning that both its real and imaginary parts are harmonic, but they do not necessarily satisfy the Cauchy-Riemann equations. In Chapter 4 we will introduce the following notion: f will be called a Bloch-type harmonic mapping if its Jacobian satisfies

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\sqrt{|J_f(z)|} < \infty.$$

This gives rise to a new class of mappings which generalizes and contains the well-known analytic Bloch space. It will be seen that this class is affine and Möbius invariant, though it is not a linear space. Our definition is also more general than that of Colonna [19] who considered harmonic mappings that are Lipschitz between the unit disk $\mathbb{D}$ endowed with the hyperbolic metric and $\mathbb{C}$ endowed with the euclidean metric.

We will give estimates for the radius of univalence, the growth and the coefficients of functions in this class. We will establish an analogue of the theorem which states that an analytic function φ is Bloch if and only if there exist a number $\alpha \in \mathbb{C}$ and a univalent function ψ such that $\varphi = \alpha \log \psi'$.

Introducción

En esta tesis se tratan algunos problemas extremales en la Teoría Geométrica de Funciones, principalmente en relación con las funciones univalentes, pero también relacionados con otras funciones analíticas o armónicas en el disco unidad $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Ha sido significativa para el desarrollo de la teoría la conjetura de Bieberbach, según la cual los coeficientes de las funciones univalentes y normalizadas deben satisfacer $|a_n| \leq n$. La conjetura fue anunciada en 1916 y probada por de Branges en 1984, dejando atrás un arsenal de técnicas, como la teoría de Loewner y una plétora de nuevos métodos variacionales, entre otros. Nuestro objetivo será estudiar dos conjeturas relacionadas con la de Bieberbach, en concreto, las conjeturas de Zalcman y de Bombieri, en los Capítulos 2 y 3, respectivamente. También está relacionada con las funciones univalentes la clase de Carathéodory que consiste en funciones con parte real positiva. En el Capítulo 1 presentaremos algunos resultados sobre los coeficientes de las funciones en esta clase.

Desde los años 80, los investigadores interesados en las funciones univalentes trataron de extender esta teoría a las aplicaciones armónicas, especialmente después de un influyente artículo de Clunie y Sheil-Small [18]. En el Capítulo 4 discutiremos una nueva definición de las aplicaciones armónicas de tipo Bloch y algunos resultados relacionados.

Las desigualdades de Livingston. En el primer capítulo nos interesarán dos desigualdades demostradas por Livingston [55, 56] para funciones en la clase $\mathcal{P}$, es decir, funciones analíticas

$$p(z) \;=\; 1 + p_1 z + p_2 z^2 + \dots, \qquad z \in \mathbb{D},$$

que satisfacen $\operatorname{Re} p(z) > 0$ para cada $z \in \mathbb{D}$. Un teorema bien conocido de

Carathéodory afirma que $|p_n| \leq 2$ para todo $n \geq 1$, mientras que el primero de los resultados de Livingston afirma que $|p_n - p_k p_{n-k}| \leq 2$ para cada $0 \leq k \leq n-1$. Veremos que esto puede generalizarse a

$$|p_n - wp_k p_{n-k}| \leq 2\,\text{máx}\{1, |1-2w|\}, \quad w \in \mathbb{C}. \tag{2}$$

También veremos una caracterización del caso de igualdad para cada $w \in \mathbb{C}$ que parece ser nueva incluso para el caso $w = 1$. Usando un lema simple probaremos que las infinitas condiciones de (2) son equivalentes a la única condición

$$\left| p_n - \frac{p_k p_{n-k}}{2} \right| \leq 2 - \frac{|p_k p_{n-k}|}{2}.$$

Por otra parte, el segundo resultado de Livingston da la cota precisa de algunos funcionales relacionados con determinantes de matrices de Hessenberg (casi-triangulares) cuyas entradas son los coeficientes p_n. Ambos resultados de Livingston se demostrarán de una manera sustancialmente más sencilla. La desigualdad (2) también nos dará una prueba nueva y más simple de un resultado de Brown [12], según el cual

$$|e^{i\nu} p_{n+m} - p_n| \leq 2\sqrt{2 - \text{Re}\,(e^{i\nu} p_m)}$$

para todo $n, m \geq 1$, $\nu \in \mathbb{R}$. Daremos algunas aplicaciones a las transformaciones del disco unidad, aunque las aplicaciones principales -y fuente de motivación- serán presentadas en el Capítulo 2 que está relacionado con la conjetura de Zalcman.

La conjetura de Zalcman. Para funciones analíticas univalentes y normalizadas

$$f(z) = z + a_2 z^2 + a_3 z^3 \ldots, \qquad z \in \mathbb{D},$$

es decir, funciones en la bien conocida clase S, la conjetura de Zalcman dice que la desigualdad

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2$$

debe ser cierta para cada $n \geq 2$. Su importancia viene del hecho de que, si se asume que es verdadera, implica la conjetura de Bieberbach (véase [13]). Se sabe que es cierta para los valores pequeños de n y varias subclases de S (véase [32] y las referencias en el mismo). Brown y Tsao [13] y Ma [58] propusieron el estudio del funcional generalizado de Zalcman $\Phi(f) = \lambda a_n a_m - a_{n+m-1}$, para $\lambda > 0$. En el Capítulo 2 continuaremos este estudio y, usando los resultados del Capítulo 1, lo extenderemos a valores complejos del parámetro λ. Obtendremos estimaciones precisas en las clausuras de las envolturas convexas de las clases de funciones convexas y estrelladas de un orden dado -dos clases que también contienen funciones no univalentes- así como para las clases de Hurwitz y Noshiro-Warschawksi. Por ejemplo, aplicando la desigualdad (2) obtendremos el siguiente resultado: Si f está en la clausura de la envoltura convexa de la clase de funciones estrelladas entonces para cada $m, n \geq 2$ tenemos que

$$|\lambda a_m a_n - a_{m+n-1}| \leq (m+n-1) \max\left\{1, \left|1 - \frac{mn}{m+n-1}\lambda\right|\right\}, \quad \lambda \in \mathbb{C}\,.$$

Esto responde a una pregunta de Ma [58] sobre el mínimo valor λ positivo para el cual una de sus estimaciones sigue siendo cierta. El mérito de considerar parámetros complejos λ, aparte de la generalización, es que esta cantidad infinita de desigualdades es equivalente a la única desigualdad

$$\left|\frac{a_m a_n}{mn} - \frac{a_{m+n-1}}{m+n-1}\right| + \frac{|a_m a_n|}{mn} \leq 1\,,$$

que tiene cierto interés en sí misma. De la misma manera, cada uno de nuestros resultados que se formula como una desigualdad que es cierta para todo $\lambda \in \mathbb{C}$ se enunciará de una manera equivalente como una sola nueva desigualdad. Observaremos que nuestro teorema para la clase de Hurwitz refleja un nuevo fenómeno para el funcional generalizado $\lambda a_m a_n - a_{m+n-1}$: las cotas precisas obtenidas difieren de manera esencial en el caso $m \neq n$ del caso $m = n$. Mostraremos que la conjetura generalizada de Zalcman es asintóticamente cierta para cada valor complejo de λ y también es equivalente a otras proposiciones relacionadas que podrían dar lugar a una mayor comprensión del problema. Mejoraremos la observación de que la conjetura de Zalcman implica la conjetura de Bieberbach mostrando que esta implicación pasa por tres conjeturas relacionadas pero más débiles que la de Zalcman.

La conjetura de Bombieri. Mucho antes de la solución final de la conjetura de Bieberbach por de Branges, muchos investigadores concentraron sus esfuerzos en resolver un problema más débil: la función de Koebe

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

debe ser, al menos, un máximo *local* en S para el funcional $\operatorname{Re} a_n$. Métodos potentes fueron empleados por Duren, Garabedian, Ross y Schiffer, entre otros, pero el problema finalmente cedió ante Bombieri [8] en 1967, que combinó la teoría de Loewner con las técnicas variacionales de Duren y Schiffer [26].

En el mismo artículo [8], Bombieri conjeturó en términos muy precisos cuál sería el comportamiento de los coeficientes de las funciones univalentes cerca de la función de Koebe. En concreto, propuso que para cualquier f en S los dos números reales

$$\sigma_{mn} := \liminf_{f \to K} \frac{n - \operatorname{Re} a_n}{m - \operatorname{Re} a_m} \qquad \text{y} \qquad B_{mn} := \min_{t \in \mathbb{R}} \frac{n \sin t - \sin(nt)}{m \sin t - \sin(mt)}$$

deben ser iguales para cada $m, n \geq 2$. Aunque se ha probado que $0 \leq \sigma_{mn} \leq B_{mn}$ y que $\sigma_{mn} = B_{mn}$ para funciones con coeficientes reales (véase [64]), la conjetura fue refutada por Greiner y Roth [40] en el caso $(m, n) = (3, 2)$ y, unos años después, Prokhorov y Vasil'ev [65] probaron que la conjetura es falsa también para los puntos $(2, 4), (3, 4)$ y $(4, 2)$.

Recientemente, Leung [51] utilizó la segunda fórmula de variación de Bombieri para probar que la conjetura es falsa para $n = 2$ y para todo $m \geq 3$ y, también, para $n = 3$ y para todo $m \geq 5$ impar. Complementando su trabajo veremos que la conjetura de Bombieri es falsa cuando $m > n \geq 2$ son simultáneamente impares o pares y, también, para el caso en el que m es impar, n es par y $n \leq \frac{m+1}{2}$. Utilizaremos sobre todo trigonometría, pero también emplearemos el criterio de Dieudonné para la univalencia de polinomios.

Aplicaciones armónicas de tipo Bloch. Sea f una aplicación armónica compleja, definida en el disco unidad $\mathbb{D}$, con el cual entendemos que ambas de sus partes real e imaginaria son armónicas, pero no necesariamente satisfacen las ecuaciones de Cauchy-Riemann. En el Capítulo 4 introduciremos la siguiente noción: f se llamará una aplicación armónica de tipo Bloch si su Jacobiano satisface

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\sqrt{|J_f(z)|}<\infty.$$

Esto da lugar a una nueva clase de funciones que generaliza y contiene el bien conocido espacio analítico de Bloch. Se verá que esta clase es invariante respecto a transformaciones afines y de Möbius, aunque no es un espacio lineal. Nuestra definición es también más general que la de Colonna [19] que consideró las aplicaciones armónicas que son Lipschitz entre el disco unidad $\mathbb{D}$ con la métrica hiperbólica y $\mathbb{C}$ con la métrica euclídea.

Daremos estimaciones para el radio de univalencia, el crecimiento y los coeficientes de funciones en esta clase. Estableceremos un análogo del teorema que dice que una función analítica φ es Bloch si y sólo si existe un número $\alpha\in\mathbb{C}$ y una función univalente ψ tal que $\varphi=\alpha\log\psi'$.

Acknowledgments

I would like to express my deep gratitude to all of the people who throughout the years have helped, taught, influenced and inspired me. It is difficult to include everyone in these few lines and so, unavoidably, some will be left out. To these people I ask for an apology.

To my beloved teacher from my school years Eleni Mitsiou, I am grateful for the many hours of preparation for the mathematics competitions, through which my enthusiasm for mathematics was greatly strengthened. I am grateful to the Department of Mathematics at the Aristotle University of Thessaloniki for the comprehensive undergraduate preparation it provided. In particular, I would like to warmly thank professor Dimitrios Betsakos for introducing me to Complex Analysis and supervising my Master's thesis. In many aspects his way of thinking about mathematics will always be with me.

I am greatly indebted to Universidad Autónoma de Madrid for the fellowship from the International Excellence Graduate Program in Mathematics that made my doctoral studies possible. I also acknowledge the partial support from the grant MTM2015-65792-P by MINECO. I am grateful to the faculty of the Department of Mathematics in UAM for the excellent working atmosphere they provide. In particular, I would like to thank professors Daniel Faraco, José Luis Fernández and Alberto Ruiz for their teaching. I specially want to thank professor Dmitry Yakubovich for his insightful questions in the seminars and for suggesting that a simple proposition such as the lemma we present in Section 1.2 could be true and helpful in our analysis.

During my graduate studies I had the opportunity to travel to some places. Here I would like to take a moment to thank my hosts. I am grateful to professor Oliver Roth for his hospitality at the University of Würzburg, Germany. Both of my stays there have been very fruitful. There I have also met with professor Yuk J. Leung to whom I am especially indebted for sharing with me his ideas on variational methods and, also, for making the crucial suggestion for what is now the main theorem in Chapter 3. I would also like to thank professor María Martín for her hospitality at the University of Joensuu, Finland, and for her valuable comments on the content of this thesis. I give many thanks to professors Rodrigo Hernández and Martin Chuaqui for their hospitality in Chile. I look forward to future collaborations with them, since I write these lines only a few days after accepting a post-doctoral position in Santiago, Chile.

The doctoral students in the Department form exceptionally strong bonds and friendships. I thank all of them for the good times we have had. I have greatly enjoyed our discussions with my mathematical sister Irina Arévalo. I always felt at home with my office-mate and fellow musician Beatriz. I thank Tania for co-organizing the Seminario Junior with me. I thank all of my friends Alberto, Alejandro, Bea, Carlos, Cristóbal, Dani, David, Francisco, Glenier, Jaime, Javi, Juan, Marcos, Marta, Martí, Nikita, Raquel and Sauli, and all the others who have passed through the Department in the recent years. A special thanks goes to my roommate Leyter, for putting up with the sounds of scales and the metronome.

I am very grateful to Bobby Martínez for having me in his big band in the AUX University and for being an inspiration. Because of him I am now less ignorant on Jazz than I used to be.

Most of all, I want to express my gratitude to my advisor, professor Dragan Vukotić, for his encouragement and help throughout my doctoral studies. I greatly appreciate the time that he devoted to me and his patience in carefully proofreading my manuscripts. His insight and enthusiasm made it a joy to learn from him. I am proud to be one of his students.

Finally, I thank my parents Dimitrios and Despina for providing me with all the opportunities one needs to become a mathematician. Becoming one is and will be a responsibility that is only mine.

List of symbols

$\mathbb{N}$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$
$\mathbb{T}$	the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : \|z\| = 1\}$
$\mathbb{A}$	the exterior of the unit disk $\{z \in \mathbb{C} : \|z\| > 1\}$
$H(\mathbb{D})$	the space of analytic functions in $\mathbb{D}$
$\mathcal{P}$	the Carathéodory class of functions
S	the class of normalized univalent functions in $\mathbb{D}$
Σ	the class of normalized univalent functions in $\mathbb{A}$
Σ'	the class of non-vanishing functions in Σ
$\widetilde{\Sigma}$	the class of full mappings in Σ
C	the class of convex functions in S
$C(\alpha)$	the class of convex functions of order α
S^*	the class of starlike functions in S
$S^*(\alpha)$	the class of starlike functions of order α
$\mathcal{H}$	the Hurwitz class
$\mathcal{R}$	the Noshiro-Warschawski class
$\mathcal{B}$	the Bloch space
$d_f(z)$	the radius of univalence of a mapping f at a point z
$d_h(z, w)$	the hyperbolic distance between z and w in $\mathbb{D}$
Δ	the Laplace operator

Chapter 0

Preliminaries

In this chapter we discuss analytic functions with positive real part, along with their representation through the Herglotz formula and the theorems of Carathéodory and Toeplitz about their coefficients. Then we present some of the theory of univalent functions in the unit disk and its exterior, including the area theorem, Dieudonné's criterion and Hayman's regularity theorem. We consider four special classes of univalent functions. These are the classes of Noshiro-Warschawski, Hurwitz, convex and starlike functions. We give a brief account of Loewner's parametric method and two of its applications. We then present the space of Bloch functions and some of its connections with univalent functions and the radius of univalence. Finally, we discuss harmonic mappings and the pre-Schwarzian derivative.

0.1 Functions with positive real part

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, $\mathbb{T} = \partial\mathbb{D}$ the unit circle and $H(\mathbb{D})$ the space of analytic functions in $\mathbb{D}$. Let

$$\mathcal{P} \;=\; \{p \in H(\mathbb{D}) \,:\, p(0) = 1,\, \operatorname{Re} p(z) > 0 \text{ for all } z \in \mathbb{D}\}$$

be the *Carathéodory class* of functions. A typical example of a function in $\mathcal{P}$ is the function

$$\frac{1+z}{1-z} \;=\; 1 + 2z + 2z^2 + \dots, \tag{1}$$

which maps $\mathbb{D}$ conformally onto the right half-plane. According to Carathéodory's inequality the coefficients of functions

$$p(z) \;=\; 1 + p_1 z + p_2 z^2 + \ldots \tag{2}$$

in $\mathcal{P}$ must satisfy $|p_n| \le 2$. Equality holds for the half-plane function (1) but, as we will see in Theorem 0.1, there are many more extremal functions.

Two simple transformations that preserve the class $\mathcal{P}$ are the *inversion*

$$q(z) \;=\; \frac{1}{p(z)} \;=\; 1 - p_1 z - (p_2 - p_1^2)z^2 + \ldots$$

and the *rotation*

$$q(z) \;=\; p(\lambda z), \quad \lambda \in \mathbb{T},$$

whose n-th coefficient is $q_n = \lambda^n p_n$. Observe that the functional $p_2 - p_1^2$ appears as the second coefficient of the inversion. Expressions like this will be our central theme in Chapters 1 and 2.

Herglotz representation. The following integral representation for the class $\mathcal{P}$ is of great importance in many situations in geometric function theory. According to the *Herglotz representation*, for any $p \in \mathcal{P}$ there exists a unique probability measure μ supported on $\mathbb{T}$ such that

$$p(z) = \int_{\mathbb{T}} \frac{1+\lambda z}{1-\lambda z} d\mu(\lambda), \quad z \in \mathbb{D}. \tag{3}$$

We call μ the *Herglotz measure* of p and write *supp*(μ) for its support. One can readily see that the coefficients of p satisfy

$$p_n = 2\int_{\mathbb{T}} \lambda^n d\mu(\lambda). \tag{4}$$

A proof of (3) can be given via the Poisson formula and Helly's selection theorem (see [23, §1.9]). Alternatively, the Herglotz representation can be deduced from the Krein-Milman theorem (see [67, Chapter 1]) once the extreme points of $\mathcal{P}$ have been determined. These were found by Holland [45] (and later by Kortram [48] with a different proof) to be the rotations of the half-plane function (1), that is, the set of functions $\{\frac{1+\lambda z}{1-\lambda z} : \lambda \in \mathbb{T}\}$.

Carathéodory's theorem. The following theorem of Carathéodory [16] (see also [62, §2.1] or [23, §2.5]) provides precise information about the Taylor coefficients of functions in the class $\mathcal{P}$. We will need the following notation. For any $n \in \mathbb{N}$ let

$$U_n \;=\; \{e^{2k\pi i/n} : k = 1, 2, \ldots, n\}$$

be the set of n-th roots of unity. For $n = 0$ we understand U_0 as $\mathbb{T}$. Also, for a set $E \subset \mathbb{C}$ and a number $a \in \mathbb{C}$ we write $aE = \{az : z \in E\}$.

Theorem 0.1 (Carathéodory)**.** *If $p \in \mathcal{P}$ then $|p_n| \leq 2$ for all $n \geq 1$. For a fixed n, equality holds if and only if supp(μ) $\subseteq e^{i\theta}U_n$ for some $\theta \in [0, 2\pi)$.*

We note that if $|p_n| = 2$ then the form of the extremal functions in this theorem is

$$p(z) \;=\; \sum_{k=0}^{n} m_k \frac{1 + \lambda_k z}{1 - \lambda_k z}\,,$$

where

$$\lambda_k = e^{i\left(\theta + \frac{2k\pi}{n}\right)}, \quad m_k \in [0, 1] \quad \text{and} \quad \sum_{k=1}^{n} m_k = 1.$$

The modern one-line proof of Theorem 0.1 makes use of the Heglotz representation: apply the triangle inequality at (4) and see that equality is possible only if λ^n has constant argument on the support of the measure μ.

Carathéodory-Toeplitz theorem. To end this section we discuss a theorem of Carathéodory and Toeplitz[1] from 1911 which characterizes the class $\mathcal{P}$ in terms of Taylor coefficients. Although we will not use it directly, we will compare it to some of our results. It can be found in [70, Ch.IV, §7] or [39, Ch.9], for example. First, for any analytic function of the form (2) we define the following symmetric matrix with constant entries on all

[1]This theorem combined with the Herglotz formula provides a solution to the trigonometric moment problem, which asks whether a given sequence of complex numbers consists of the moments of some Borel measure supported on the unit circle (see [39, Ch.4]).

diagonals parallel to the main one:

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_{n-1} & p_n \\ \overline{p}_1 & 2 & p_1 & \cdots & p_{n-2} & p_{n-1} \\ \overline{p}_2 & \overline{p}_1 & 2 & \cdots & p_{n-3} & p_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{p}_{n-1} & \overline{p}_{n-2} & \overline{p}_{n-3} & \cdots & 2 & p_1 \\ \overline{p}_n & \overline{p}_{n-1} & \overline{p}_{n-2} & \cdots & \overline{p}_1 & 2 \end{vmatrix}$$

Theorem 0.2 (Carathéodory-Toeplitz)**.** *Let $p \in H(\mathbb{D})$ have the form* (2)*. Then $p \in \mathcal{P}$ if and only if $D_n \geq 0$ for all $n \geq 1$.*

A few examples of the condition $D_n \geq 0$ for some initial n are the following: for $n = 1$ we have $|p_1| \leq 2$, for $n = 2$

$$2|p_1|^2 + |p_2|^2 \leq 4 + \mathrm{Re}\,(p_1^2\overline{p_2})$$

and for $n = 3$

$$\begin{aligned} &12|p_1|^2 + 8|p_2|^2 + 4|p_3|^2 + 2|p_1p_2|^2 + 2\mathrm{Re}\,(p_1^3\overline{p_3}) + 2\mathrm{Re}\,(p_1\overline{p_2}^2p_3) \\ \leq\, &16 + |p_1|^4 + |p_2|^4 + |p_1p_3|^2 + 8\mathrm{Re}\,(p_1^2\overline{p_2}) + 8\mathrm{Re}\,(p_1p_2\overline{p_3}). \end{aligned}$$

Due to the increasing complexity of these inequalities, they are of little practical help in coefficient problems for general n and, thus, alternatives are needed.

0.2 Basic theory of univalent functions

A function is said to be *univalent* in a complex domain if it is analytic and injective there. As is customary, we denote by S the class of *normalized* univalent functions in the unit disk with the expansion

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots, \qquad z \in \mathbb{D}.$$

An important member of S is the *Koebe* function

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$

The class S is preserved under the *rotation* transformation $f_\lambda(z) = \overline{\lambda} f(\lambda z)$, $\lambda \in \mathbb{T}$, whose n-th coefficient is $\lambda^{n-1} a_n$. For any function in S the estimate $|a_2| \le 2$ holds by Bieberbach's celebrated theorem (1916). Many important theorems can be deduced from this: for example, the Koebe one-quarter theorem, which states that the image of $\mathbb{D}$ under any function in S covers the open disk $|w| < 1/4$ (see [23, §2.2]). Also, Bieberbach's theorem implies that any $f \in S$ satisfies the condition

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \le \frac{4r}{1-r^2}, \qquad r = |z| < 1. \tag{5}$$

This inequality implies the well-known growth and distortion estimates in the class S:

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \qquad r = |z| < 1 \tag{6}$$

and

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, \qquad r = |z| < 1,$$

in both of which equality holds for some $z \neq 0$ only for a suitable rotation of the Koebe function (see [23, §2.3]).

Throughout the long history of the theory of univalent functions one of the motivating forces has been the *Bieberbach conjecture* which says that $|a_n| \le n$ for all $n \ge 2$ and that the only extremal function should be the Koebe function. This long-standing problem was finally solved by L. de Branges in 1984 (see [9] or [42] for a proof).

Area Theorem. Let $\mathbb{A} = \{z \in \mathbb{C} : |z| > 1\}$ denote the *exterior* of the unit disk. Let Σ denote the class of univalent functions in $\mathbb{A}$ which have a simple pole at $z = \infty$ with residue equal to 1. Functions in Σ have the expansion

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots, \qquad z \in \mathbb{A}.$$

An example is the *Joukowsky* function $z + \frac{1}{z}$, which maps $\mathbb{A}$ to the complement of the interval $[-2, 2]$. We say that a function g in Σ is a *full mapping* if the two-dimensional Lebesgue measure of $\overline{\mathbb{C}} \backslash g(\mathbb{A})$ is zero. We denote by $\widetilde{\Sigma}$ the set of full mappings and by Σ' the set of functions in Σ that do not vanish. A connection between univalent functions in $\mathbb{D}$ and $\mathbb{A}$ comes from the *inversion* transformation as follows: $f \in S$ if and only if

$$g(z) \;=\; \frac{1}{f(1/z)} \;=\; z - a_2 + \frac{a_2^2 - a_3}{z} + \dots \tag{7}$$

is in Σ'.

The following theorem of Gronwall (1914) is of great value in the theory of univalent functions and has acquired vast generalizations. Its standard proof involves contour integration and makes use of Green's theorem (see [23, §2.2]).

Theorem 0.3 (Area Theorem)**.** *If $g \in \Sigma$ then $\sum_{n=1}^{\infty} n|b_n|^2 \le 1$. Equality holds if and only if $g \in \widetilde{\Sigma}$.*

The following corollary is well-known and appears as Exercise 1 in [23, Ch.2] and, also, as Problem 2 in [62, §1.2]. Here we include a proof in order to clarify the case of equality.

Corollary 0.4. *If $f \in S$ then $|a_2^2 - a_3| \le 1$. Equality holds if and only if f is a rotation of the function*

$$\frac{z}{1 + xz + z^2} \;=\; z - xz^2 + (x^2 - 1)z^3 + \dots$$

for some $x \in [-2, 2]$.

Proof. Since the area theorem implies that $|b_1| \le 1$, we get the desired inequality from the inversion formula (7).

If equality holds then $b_1 = \lambda^2$ for some $\lambda \in \mathbb{T}$ and also, all the rest of the coefficients of g vanish due to the area theorem. Hence g has the form

$$g(z) \;=\; z + b_0 + \frac{\lambda^2}{z} \;=\; \lambda\left(\overline{\lambda} z + \frac{1}{\overline{\lambda} z}\right) + b_0,$$

which is a rotation and translation of the Joukowski function. But g does not vanish and therefore $b_0 = x\lambda$ for some $x \in [-2, 2]$. It follows that

$$f(z) \;=\; \frac{1}{g(1/z)} \;=\; \frac{z}{1 + \lambda x z + \lambda^2 z^2}.$$

$\square$

The range of the extremal function in Corollary 0.4 is the complement of two radial slits emanating from the points

$$\frac{\overline{\lambda}}{x-2} \qquad \text{and} \qquad \frac{\overline{\lambda}}{x+2}$$

for some $x \in [-2, 2]$ and some $\lambda \in \mathbb{T}$. In the limiting cases $x = \pm 2$ we understand that there is only one slit and therefore the extremal function is a rotation of the Koebe function.

Dieudonné's criterion. An analytic function f is locally univalent if and only if $f'(z) \neq 0$ for all $z \in \mathbb{D}$. Obviously, all univalent functions are locally univalent. The following elementary lemma [22] (see also [23, p.75]) shows that for polynomials the condition $f' \neq 0$ to be satisfied by a generalization of the derivative can be necessary and sufficient for univalence.

Lemma 0.5 (Dieudonné)**.** *The polynomial $p(z) = z + a_2 z^2 + \ldots + a_n z^n$ is univalent in $\mathbb{D}$ if and only if its associated polynomials*

$$q(z;t) \;=\; 1 + a_2 \frac{\sin(2t)}{\sin t} z + \ldots + a_n \frac{\sin(nt)}{\sin t} z^{n-1}$$

have no zeros in $\mathbb{D}$ for any choice of the parameter $t \in [0, \pi]$.

Note that for $t = 0$ we recover the derivative of p, that is, $q(z; 0) = p'(z)$. Also, the unit disk plays no special role here; the lemma can be effectively stated for $|z| < r$ instead of $\mathbb{D}$ for any $r > 0$.

Hayman's regularity theorem. For any f analytic in $\mathbb{D}$ we denote by $M_\infty(r, f)$ the maximum of the modulus of f on the circle $|z| = r$. The *Hayman index* of a function $f \in S$ is defined as the number

$$\alpha = \lim_{r \to 1^-} (1 - r)^2 M_\infty(r, f)$$

(see [23, §5.5]). Of course $\alpha \geq 0$. It is well known that $\alpha \leq 1$ and that $\alpha = 1$ only for the Koebe function and its rotations. Also, if $\alpha > 0$ then there is a unique *direction of maximal growth* $e^{i\vartheta_0}$, that is,

$$\lim_{r \to 1^-} (1 - r)^2 |f(re^{i\vartheta_0})| = \alpha.$$

The following remarkable theorem shows that asymptotically the coefficients of any function in S behave in a regular way, that is, as a constant multiple of n and that the constant is precisely the Hayman index (see [23, §5.7]).

Theorem 0.6 (Hayman). *If $f \in S$ then* $\lim_{n\to\infty} |a_n|/n = \alpha$.

0.3 Special classes of univalent functions

The Noshiro-Warschawski class. Let

$$\mathcal{R} = \{f \in \mathcal{H}(\mathbb{D}) \;:\; \mathrm{Re} f'(z) > 0,\ f(0) = f'(0) - 1 = 0\}$$

be the *Noshiro-Warschawski* class. A typical example of a function in $\mathcal{R}$ is

$$f(z) \;=\; 2\log\frac{1}{1-z} - z \;=\; z + \sum_{n=2}^{\infty} \frac{2}{n} z^n, \tag{8}$$

whose derivative is $f'(z) = (1+z)/(1-z)$, a function in the class $\mathcal{P}$. The branch of the logarithm is chosen so that $\log 1 = 0$.

Note that $\mathcal{R} \subset S$ by the basic Noshiro-Warschawski lemma [23, §2.6]. MacGregor [59] showed that for f in $\mathcal{R}$ we have $|a_n| \le 2/n$.

The Hurwitz class. The *Hurwitz* class $\mathcal{H}$ consists of all normalized functions $f \in H(\mathbb{D})$ which have the property that

$$\sum_{n=2}^{\infty} n|a_n| \;\le\; 1.$$

Obviously, the n-th coefficient of a function in $\mathcal{H}$ is subject to the estimate $|a_n| \le 1/n$ for each n. The simplest example of a function in $\mathcal{H}$ is the polynomial $z + \frac{z^n}{n}$, $n \ge 2$.

It is a well-known exercise that $\mathcal{H} \subset S$. In fact, a stronger inclusion is true: $\mathcal{H} \subset \mathcal{R}$. This can be seen as follows. If f is a function in $\mathcal{H}$ other than the identity, then $f'(0) = 1$ and, when $z \ne 0$, we have the strict inequality

$$\mathrm{Re}\, f'(z) = 1 + \sum_{n=2}^{\infty} n\mathrm{Re}\,\{a_n z^{n-1}\} \ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \ge 0\,.$$

The reader is referred to [36] for further properties of $\mathcal{H}$.

Convex functions. We now turn to the class of normalized *convex* functions in S, which we will denote by C. A typical example is the half-plane function $\ell(z) = \frac{z}{1-z}$. According to a theorem of Loewner the coefficients of functions in C must satisfy $|a_n| \leq 1$, with equality only for the function ℓ and its rotations [23, §2.5].

The analytic description of functions f in C makes use of two basic facts. First, convexity is a hereditary property, *i.e.*, $\{f(z) : |z| \leq r\}$ is convex for any $r \in (0,1)$ and, second, the slope of the tangent of $\{f(z) : |z| = r\}$ is non-decreasing as the curve is traversed in the positive direction. Thus for normalized $f \in H(\mathbb{D})$ we have that $f \in C$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We denote by $\operatorname{co}(C)$ the convex hull of C and by $\overline{\operatorname{co}}(C)$ its closure in the topology of uniform convergence on compact subsets of $\mathbb{D}$. Note that this larger class no longer consists exclusively of univalent functions. A well-known result of Marx and Strohhäcker [62, p.45] states that

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2}, \qquad z \in \mathbb{D},$$

for all functions in C. As observed in [10], this implies that

$$\overline{\operatorname{co}}(C) = \big\{f \in H(\mathbb{D}) \,:\, \operatorname{Re}\big(f(z)/z\big) > 1/2,\ f(0) = f'(0) - 1 = 0\big\}.$$

Thus, a connection with the class $\mathcal{P}$ is readily established by the formula

$$2f(z) \;=\; z\big(p(z) + 1\big), \tag{9}$$

that is, $f \in \overline{\operatorname{co}}(C)$ if and only if f can be written as in (9) for some $p \in \mathcal{P}$.

Starlike functions. A set E is said to be *starlike* (with respect to the origin) if for every $z \in E$ the entire segment $[0, z]$ is contained in E. The class S^* of starlike functions consists of normalized univalent functions whose image is a starlike domain. Obviously, $C \subset S^* \subset S$.

In a complete analogy with the class C, the analytic description of S^* relies on the fact that starlikeness is a hereditary property and on the geometric property that the curves $\{f(z) : |z| = r\}$ possess. The latter is in this case even simpler: the argument of $f(z)$ in non-decreasing as $|z| = r$

is traversed in the positive direction. Hence, for normalized $f \in H(\mathbb{D})$ we have that $f \in S^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Now, a theorem of Alexander [23, §2.5] connects the two classes via their analytic characterizations. It states that for any normalized $f \in H(\mathbb{D})$ we have that

$$f \in C \quad \text{if and only if} \quad zf'(z) \in S^*.$$

It simply amounts to writing $g = zf'$ and noting that

$$\frac{zg'}{g} = 1 + \frac{zf''}{f'}.$$

Alexander's relation is preserved upon taking convex combinations and uniform limits on compact subsets of the disk. Hence

$$f \in \overline{\mathrm{co}}(C) \quad \text{if and only if} \quad zf'(z) \in \overline{\mathrm{co}}(S^*).$$

In view of (9) we find a formula connecting $\overline{\mathrm{co}}(S^*)$ to the class $\mathcal{P}$:

$$g(z) = zf'(z) = z\left(\frac{zp(z)+z}{2}\right)' = \frac{z}{2}\big(1 + p(z) + zp'(z)\big), \tag{10}$$

that is, $g \in \overline{\mathrm{co}}(S^*)$ if and only if g can be written as in (10) for some $p \in \mathcal{P}$.

Convex and starlike functions of order α. To end this section we present a parametrized version of the last two classes that we have discussed. We say that a normalized $f \in H(\mathbb{D})$ is *convex* or, respectively, *starlike of order* α if

$$\operatorname{Re}\left(1 + \frac{zf''}{f'}\right) > \alpha \qquad \text{or} \qquad \operatorname{Re}\left(\frac{zf'}{f}\right) > \alpha,$$

and denote the respective classes by $C(\alpha)$ and $S^*(\alpha)$. Both classes were introduced by Robertson in [66, §3] (see also [38, §2.3] and [69, §5]). The definitions make sense for any $\alpha \leq 1$, while for $\alpha = 1$ each of the classes contains only the identity mapping. Obviously these families of classes are nested and decreasing in α. For $\alpha = 0$ we recover the known class $C = C(0)$ and therefore all functions in $C(\alpha)$ are univalent and convex whenever

$0 \le \alpha \le 1$. The analogue is true for $S^*(\alpha)$. Robertson [66] observed that functions f in $C(\alpha)$, $\alpha \in [0,1]$, have the following geometric property: the ratio of the angle between adjacent tangents of the unit circle over the angle between the corresponding tangents in the image of f is less than $1/\alpha$. Hence the closer α is to 1 the "rounder" is the image. For instance, segments in the boundary of $f(\mathbb{D})$ are prohibited as soon as $\alpha > 0$.

Umezawa [71] showed that all functions in $C(-1/2)$ are univalent and convex in one direction. (We mention that a domain is said to be convex in the direction $e^{i\theta}$ if its intersection with each line parallel to the line passing through the origin and $e^{i\theta}$ is connected or empty.) We shall see that $\alpha = -1/2$ is the smallest α for which $C(\alpha)$ consists only of univalent functions. Also, $S^*(\alpha)$ contains non-univalent functions for $\alpha < 0$.

Alexander's theorem remains true in this setting and has the form

$$f \in C(\alpha) \quad \text{if and only if} \quad zf'(z) \in S^*(\alpha).$$

The function given by

$$f_\alpha(z) \;=\; z + \sum_{n=2}^{\infty} A_n z^n \;=\; \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \text{for } \alpha \neq 1/2, \\ \log \frac{1}{1-z}, & \text{for } \alpha = 1/2, \end{cases}$$

is often extremal in $C(\alpha)$. Its coefficients are given by

$$A_n = \frac{\Gamma(n+1-2\alpha)}{n!\,\Gamma(2-2\alpha)} \;=\; \frac{1}{n!} \prod_{k=2}^{n} (k-2\alpha).$$

In $S^*(\alpha)$ the typical example is

$$g_\alpha(z) \;=\; z + \sum_{n=2}^{\infty} B_n z^n \;=\; \frac{z}{(1-z)^{2-2\alpha}},$$

whose coefficients are given by $B_n = nA_n$. Both examples f_α and g_α manifest the claim made earlier about the smallest value of α for which their respective classes consist only of univalent functions. Indeed, f_α violates the growth theorem (6) when $\alpha < -1/2$, and the same is true for g_α when $\alpha < 0$.

An integral representation was given in [11] for the closure of the convex hull of the class $S^*(\alpha)$ for any $\alpha < 1$. It states that if $g \in \overline{\mathrm{co}}\big(S^*(\alpha)\big)$ then there exists a unique probability measure μ supported on $\mathbb{T}$ such that

$$g(z) \;=\; \int_{\mathbb{T}} \frac{z}{(1-\lambda z)^{2-2\alpha}} d\mu(\lambda).$$

This readily yields a relation between the coefficients b_n of g and the coefficients p_n of some function in the Carathéodory class $\mathcal{P}$, in particular,

$$b_n \;=\; \frac{B_n\, p_{n-1}}{2}.$$

Now the estimate $|b_n| \leq B_n$ follows at once from Theorem 0.1. By Alexander's theorem, the integral representation carries over to functions f in $\overline{\mathrm{co}}\big(C(\alpha)\big)$ as

$$f(z) \;=\; \int_{\mathbb{T}} \overline{\lambda} f_\alpha(\lambda z) d\mu(\lambda).$$

Evidently, the coefficients of f satisfy

$$a_n \;=\; \frac{A_n\, p_{n-1}}{2} \tag{11}$$

for some $p \in \mathcal{P}$. Once again, Theorem 0.1 yields the estimate $|a_n| \leq A_n$.

Motivated by the Fekete-Szëgo theorem (which we discuss in the next section), Keogh and Merkes [47] proved a sharp inequality for spirallike functions of a given order. These functions are more general than functions in the class $S^*(\alpha)$ and will not be considered here. Instead we state a special case of their theorem: Any $f \in S^*(\alpha)$, for $0 \leq \alpha < 1$, satisfies

$$|a_3 - \lambda a_2^2| \;\leq\; (1-\alpha)\max\{1,\, |2(1-\alpha)(2\lambda-1)-1|\}, \qquad \lambda \in \mathbb{C}.$$

In Chapter 2 we will prove a similar inequality for the generalized Zalcman functional $a_{m+n-1} - \lambda a_m a_n$ for any $m, n \geq 2$ and $\lambda \in \mathbb{C}$. Our estimate will hold for functions in the closed convex hull of $S^*(\alpha)$ for any $\alpha < 1$.

0.4 Loewner chains and applications

The basis of Loewner's parametric method is that the image of any function in S can be embedded in a continuously increasing family of domains whose Riemann mappings can then be described by a differential equation.

A *Loewner chain* is a set of univalent functions

$$f(z,t) \;=\; e^t z + a_2(t) z^2 + \dots, \qquad z \in \mathbb{D},\ t \geq 0, \tag{12}$$

which satisfy the subordination condition

$$f(z,s) \;=\; f\big(\varphi(z,s,t),t\big), \qquad 0 \leq s \leq t < \infty, \tag{13}$$

for some univalent $\varphi(\cdot,s,t) : \mathbb{D} \to \mathbb{D}$ satisfying $|\varphi(z,s,t)| \leq |z|$. For any Loewner chain there exist functions $p(\cdot,t) \in \mathcal{P}$, measurable in $t \geq 0$, such that

$$\frac{\partial f}{\partial t}(z,t) \;=\; z\,\frac{\partial f}{\partial z}(z,t)\,p(z,t) \tag{14}$$

for almost all t (see §6.1 in [62]). This equation has a simple geometric interpretation that is visible when we take the logarithm of both sides and then consider their imaginary parts. We then have that

$$\left|\arg\left(\frac{\partial f}{\partial t}\right) - \arg\left(z\,\frac{\partial f}{\partial z}\right)\right| = |\arg\left(p(z,t)\right)| < \frac{\pi}{2},$$

since $p(z,t)$ lies in the right half-plane. This shows that the *flow* $f(z,t)$ is expanding, since for fixed $z = re^{i\theta} \in \mathbb{D}$ it has a velocity vector f_t that points outwards from the set $\{f(\zeta,t) : |\zeta| \leq r\}$. The key observation in Loewner's theory is that every function f in S is the initial value $f(z) = f(z,0)$ of some Loewner chain (12), which thus satisfies equation (14). The proof of this proposition relies on the Carathéodory kernel convergence of domains and the compactness of the class of Loewner chains.

We will confine ourselves to the special case when $p(\cdot,t)$ is a rotation of the half-plane function (1). Thus equation (14) becomes

$$\frac{\partial f}{\partial t}(z,t) \;=\; z\,\frac{\partial f}{\partial z}(z,t)\,\frac{1+\kappa(t)z}{1-\kappa(t)z}, \tag{15}$$

where the *drive function* κ has the form $\kappa(t) = e^{i\vartheta(t)}$, with ϑ being real-valued and piecewise continuous on $[0,\infty)$. This special case is related to single-slit mappings in S, that is, functions whose image is the complement of a single simple arc which extends to infinity. These mappings form a dense subclass of S and are thus very important in solving extremal problems: to estimate the maximum of any continuous functional over S it is sufficient to estimate it over this dense subclass. Any single-slit mapping can be seen -after an appropriate parametrization of the slit- as the initial value $f(z) = f(z,0)$ of a Loewner chain (12) with the geometric property that as t increases the range $f(\mathbb{D},t)$ is the complement of a slit which is being erased. The functions

$$w = \varphi(z,0,t), \qquad z \in \mathbb{D},\ t \geq 0,$$

defined by setting $s = 0$ in (13), that is,

$$f(z) = f\big(\varphi(z,0,t),t\big), \qquad t \geq 0, \tag{16}$$

are known to satisfy the *ordinary* differential equation

$$\frac{\partial\varphi}{\partial t} = -\varphi\,\frac{1+\kappa\varphi}{1-\kappa\varphi}, \tag{17}$$

(see [23, §3.3] or [35, Ch.III, §2]). Differentiating (16) with respect to t gives

$$0 = \frac{\partial f}{\partial w}\,\frac{\partial\varphi}{\partial t} + \frac{\partial f}{\partial t},$$

which, upon a simple substitution, shows that (15) and (17) are equivalent. We note that in the special case when $\kappa \equiv -1$ the unique solution of (15) is the chain $f(z,t) = e^t K(z)$, whose initial value is the Koebe function.

A theorem of Fekete and Szegö. For any $f \in S$, its square-root transform

$$g(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \ldots, \qquad z \in \mathbb{D},$$

gives rise to an odd univalent function. Since this transform can be seen to be invertible, normalized odd univalent functions are in one-to-one correspondence with functions in the class S. The initial coefficients a_n of f are related to those of g by

$$c_3 = \frac{a_2}{2} \qquad \text{and} \qquad c_5 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{4}\right).$$

All coefficients c_n are bounded by a theorem of Littlewood and Paley [23, §2.11]. In particular, by Bieberbach's inequality $|a_2| \le 2$ we have that $|c_3| \le 1$. In view of the square-root transform of the Koebe function

$$g(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \dots,$$

it became known as the Littlewood-Paley conjecture that $|c_n| \le 1$ should be true for all odd $n \ge 3$. This was quickly disproved by Fekete and Szegö [33], who used the Loewner method to prove the following sharp inequality

$$|a_3 - \lambda a_2^2| \le \begin{cases} 3 - 4\lambda, & \text{for} \quad \lambda \in (-\infty, 0], \\ 1 + 2e^{-2\lambda/(1-\lambda)}, & \text{for} \quad \lambda \in [0, 1], \\ 4\lambda - 3, & \text{for} \quad \lambda \in [1, \infty) \end{cases} \tag{18}$$

(see [23, §3.8]). Upon setting $\lambda = 1/4$ we readily get as a corollary the sharp inequality

$$|c_5| \le 1/2 + e^{-2/3} \approx 1.0134.$$

We note that Fekete and Szegö actually proved the deep part of inequality (18), that is, the estimate for the values $\lambda \in [0, 1]$. The other two estimates follow quite easily from the triangle inequality and the estimates $|a_2| \le 2$, $|a_3| \le 3$ and $|a_3 - a_2^2| \le 1$.

In Chapter 2 we will study the generalized Zalcman functional $a_{m+n-1} - \lambda a_m a_n$ for general $m, n \ge 2$ and $\lambda \in \mathbb{C}$, which can be seen as a generalization of the Fekete-Szegö functional $a_3 - \lambda a_2^2$.

Becker's criterion for univalence. Loewner's theory can be applied to prove the following theorem of Becker [5] (see also [38, §3.3.1]). If $f \in H(\mathbb{D})$, $f'(0) \ne 0$ and

$$(1 - |z|^2) \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \quad z \in \mathbb{D}, \tag{19}$$

then f is univalent. It was later proved by Becker and Pommerenke [6] that the constant 1 is sharp.

0.5 Bloch functions

Radius of univalence. For any $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$ the *radius of univalence*, or *schlicht radius*, $d_f(z)$ is defined as the radius of the largest disk that lies on a single sheet of the Riemann surface identified with $f(\mathbb{D})$ and is centered at the point $f(z)$, whenever z is not a branch point, *i.e.*, if $f'(z) \neq 0$. At a branch point of f the radius of univalence is defined as zero. As an example, for $f(z) = z^2$ we compute

$$d_f(z) \;=\; \min\{|z|^2, 1 - |z|^2\}, \quad z \in \mathbb{D}.$$

Bloch's famous theorem (1924) asserts the existence of an absolute constant $B > 0$ such that for all $f \in H(\mathbb{D})$ with the normalization $|f'(0)| = 1$ it holds that

$$B_f \;:=\; \sup_{z\in\mathbb{D}} d_f(z) \;\geq\; B.$$

Bloch functions. It was Landau's observation (1929) that for the constant $B := \inf B_f$ the infimum can be taken over the set of functions that not only have the above normalization, but also satisfy

$$\beta(f) := \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty. \tag{20}$$

Forty years later, Pommerenke [61] coined the term *Bloch function* for any $f \in H(\mathbb{D})$ that satisfies (20). The functional $\beta(\cdot)$ defines a seminorm, and the Banach space $\mathcal{B}$ of all Bloch functions equipped with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \beta(f)$ is called *Bloch space* and was first studied in [3]. (Additional information on the Bloch space can be found in [27, §2.6] and [38, Chapter 4].)

It is a simple consequence of the Schwarz Lemma, observed in [68], that every analytic function satisfies

$$d_f(z) \;\leq\; (1 - |z|^2)|f'(z)|, \qquad z \in \mathbb{D}.$$

A similar inequality in the reverse direction was also shown in [68, §31] for the case when the radius of univalence is uniformly bounded. Thus, $f \in \mathcal{B}$ if and only if $B_f < \infty$. We say that this is the *geometric* definition of $\mathcal{B}$, as opposed to the *analytic* definition (20).

Univalent functions. For univalent functions these inequalities take the simpler form

$$\frac{1}{4}(1-|z|^2)|f'(z)| \le d_f(z) \le (1-|z|^2)|f'(z)|, \qquad z \in \mathbb{D},$$

due to Koebe's 1/4-Theorem. Note that in this case $d_f(z)$ is simply the distance between $f(z)$ and the boundary of $f(\mathbb{D})$ and, therefore, $f \in \mathcal{B}$ if and only if $f(\mathbb{D})$ does not contain arbitrarily large disks. The typical Noshiro-Warschawski function (8) provides an example of a univalent Bloch function. Its unbounded range is contained in a horizontal strip.

The following theorem shows yet another close connection between Bloch functions and univalent functions (see [61] and, also, [38, §4.1]).

Theorem 0.7 (Pommerenke's Theorem [61]). *Let $f \in H(\mathbb{D})$. Then f belongs to $\mathcal{B}$ if and only if there exist a function $g \in S$ and a number $\alpha \in \mathbb{C}$ such that*

$$f(z) = \alpha \log g'(z) + f(0), \quad z \in \mathbb{D}.$$

The reverse direction follows from the fact that $\beta(\log g') \le 6$ for any $g \in S$ due to the estimate (5). The forward direction lies deeper: it relies on Becker's criterion (19). An application of it shows that the function defined by

$$g(z) = \int_{[0,z]} \exp\left(\frac{f(\zeta)-f(0)}{\beta(f)}\right) d\zeta, \quad z \in \mathbb{D},$$

is univalent when $f \in \mathcal{B}$.

Growth and coefficients of Bloch functions. A straightforward consequence of the analytic definition of the Bloch space is that all f in $\mathcal{B}$ satisfy the growth condition

$$|f(z)-f(0)| \le \frac{\beta(f)}{2} \log \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}$$

(see [63] or [20]). Noticing the *conformal invariance* of the space $\mathcal{B}$, that is, the property that $\beta(f) = \beta(f \circ \varphi_\alpha)$ for all disk automorphisms of the form

$$\varphi_\alpha(z) = \frac{\alpha+z}{1+\overline{\alpha}z}, \quad \alpha, z \in \mathbb{D},$$

one can deduce from the above growth condition that any $f \in \mathcal{B}$ satisfies

$$|f(z) - f(w)| \leq \beta(f)\, d_h(z, w), \quad z, w \in \mathbb{D}.$$

Here

$$d_h(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|}$$

is the hyperbolic distance in $\mathbb{D}$. In fact, this property characterizes the Bloch space and gives rise to what we call the *metric* definition: for f analytic in $\mathbb{D}$, $f \in \mathcal{B}$ if and only if f is Lipschitz between $\mathbb{D}$ endowed with the hyperbolic metric and $\mathbb{C}$ endowed with the euclidean metric.

Lastly, it was shown in [20] that for any $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{B}$, the estimate $|a_n| \leq \frac{2}{e}\beta(f)$ is true, and that the constant $2/e$ is best possible.

0.6 Harmonic mappings

Basic definitions. A real-valued function $u(x, y)$ is *harmonic* if it satisfies Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

A complex-valued function $f = u + iv$ defined on a domain $\Omega \subset \mathbb{C}$ is said to be a *harmonic mapping* if both u and v are harmonic. Many authors refer to injective mappings with this definition, but we shall not do that; when a harmonic mapping is univalent we will explicitly say so. Of course, any analytic function is harmonic, but a harmonic mapping is more general since it does not necessarily satisfy the Cauchy-Riemann equations. A simple example is an *affine mapping*

$$f(z) = az + b\overline{z} + c, \quad a, b, c \in \mathbb{C}.$$

When Ω is simply connected, f has a *canonical decomposition* $f = h + \overline{g}$, where h and g are analytic in Ω. The Jacobian of f is then given by $J_f = |h'|^2 - |g'|^2$ and, by a theorem of Lewy (1936), f is *locally univalent* if and only if $J_f \neq 0$. Important information about f is stored in its (*analytic*) *dilatation* $\omega = g'/h'$. For example, f is *orientation-preserving* if and only if

$|\omega(z)| < 1$ in Ω. We say that f is *orientation-reversing* if $\overline{f}$ is orientation-preserving. Naturally, a locally univalent f preserves the orientation if and only if $J_f > 0$.

A sense-preserving homeomorphism f is called *quasiconformal* if it maps infinitesimal circles onto infinitesimal ellipses having ratio of the major over the minor axis bounded from above by some constant. If f is differentiable this is equivalent to

$$D_f \;=\; \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \;\leq\; K,$$

for some $K \geq 1$. Consequently, a harmonic mapping is quasiconformal if its dilatation is bounded away from one, that is, $|\omega(z)| \leq k < 1$ in $\mathbb{D}$.

Univalent harmonic mappings. In their influential article, Clunie and Sheil-Small [18] considered *univalent* harmonic mappings in $\mathbb{D}$ and took a step in the direction of generalizing the classical theory of the class S in this context. The following are the standard normalizations. For a harmonic, univalent and sense-preserving mapping $f = h + \overline{g}$ in $\mathbb{D}$ we write

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n,$$

and say that $f \in S_H$ if it satisfies $a_0 = 1 - a_1 = 0$. We say that $f \in S_H^0$ if in addition $b_1 = 0$.

A simple use of the Schwarz Lemma [24, §5.4] yields the sharp inequality $|b_2| \leq 1/2$ for functions in S_H^0. It takes more effort to prove that

$$|a_2| \;<\; \frac{32\pi}{27}(\pi + 6\sqrt{3}) - 2 \;<\; 48.4$$

in S_H^0 [24, §6.3], and still, the best known constant 48.4 is quite distant from the conjectured 5/2.

For the larger class S_H, we have that $|b_1| < 1$ simply because f is sense-preserving. Also, it is possible to translate the preceding inequalities by means of an affine transformation. Given $f \in S_H$, the function

$$f_0 = \frac{f - \overline{b_1 f}}{1 - |b_1|^2} \tag{21}$$

belongs to S_H^0. This transformation is invertible, so that $f = f_0 + \overline{b_1 f_0}$. Hence, it is not difficult to see that

$$|a_2| < 48.4 + \frac{|b_1|}{2} \tag{22}$$

for functions in S_H.

Covering theorems. The analogue of Koebe's 1/4-theorem in this context states that the range of every mapping in the class S_H^0 contains the disk $|w| < 1/16$. It is conjectured that the constant should be 1/6 (see [24, §6.2]). Applying as before the affine transformation (21) it is easy to deduce that

$$\left\{ w \in \mathbb{C} \,:\, |w| < \frac{1 - |b_1|}{16} \right\} \subset f(\mathbb{D}) \tag{23}$$

for every $f \in S_H$ (see [18, Corollary 4.5]).

A result in the opposite direction states that each function in S_H omits some point on the circle $|w| = \frac{\pi}{2}$. In other words

$$\big(\mathbb{C} \backslash f(\mathbb{D})\big) \bigcap \left\{ w \in \mathbb{C} \,:\, |w| = \frac{\pi}{2} \right\} \neq \emptyset, \tag{24}$$

for every $f \in S_H$. The constant $\frac{\pi}{2}$ was given by Hall [41] and is best possible. See also [24, §6.2].

Schwarzian derivative. For a locally univalent analytic function φ in a simply connected domain Ω, the *pre-Schwarzian derivative* $P\varphi$ is defined as the logarithmic derivative of φ', that is,

$$P\varphi = \frac{\varphi''}{\varphi'},$$

and the *Schwarzian derivative* $S\varphi$ is defined by

$$S\varphi = (P\varphi)' - \frac{1}{2}(P\varphi)^2.$$

We have already seen the use of $P\varphi$ in Becker's univalence criterion (19). For other uses of these operators in finding criteria for univalence and for some of their many properties we refer the reader to [23, §8.5].

Let $f = h + \overline{g}$ be a locally univalent harmonic mapping in a simply connected domain Ω and let $\omega = g'/h'$ be its dilatation. In a recent work of

Hernández and Martín [44] a new Schwarzian derivative was defined by

$$S_f = Sh + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\right)^2.$$

For our purposes we will only need a pre-Schwarzian derivative that was also defined in [44] by

$$P_f = \frac{h''}{h'} - \frac{\overline{\omega}\omega'}{1-|\omega|^2}.$$

The following theorem can be seen as a harmonic analogue of Becker's criterion for univalence.

Theorem 0.8 (Harmonic Becker-Type Criterion [44]). *Let $f = h + \overline{g}$ be a sense-preserving harmonic function in the unit disk with dilatation ω. If for all $z \in \mathbb{D}$*

$$|zP_f(z)| + \frac{|z\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{1}{1-|z|^2},$$

then f is univalent.

Chapter 1

Livingston's inequalities

1.1 Statements of the results

For functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

in the Carathéodory class $\mathcal{P}$, Livingston [55] proved that

$$|p_n - p_k p_{n-k}| \le 2, \qquad 0 \le k \le n-1.$$

He used this inequality in his study of the class of multivalent close-to-convex functions, which extend the family of close-to-convex functions introduced by Kaplan [46]. More applications of Livingston's inequality were later found in [13], [54] and [58]. In [28] we provide the following generalization.

Theorem 1.1 (Complex Livingston Inequality). *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then*

$$|p_n - w p_k p_{n-k}| \le 2 \max\{1, |1-2w|\} \tag{1.1}$$

for all $1 \le k \le n-1$.

Let μ be the Herglotz measure of p. In the case $|1-2w| < 1$, equality holds if and only if $p_k = 0$ and supp$(\mu) \subseteq e^{i\varphi} U_n$ for some $\varphi \in [0, 2\pi)$. In the case $|1-2w| > 1$, equality holds if and only if supp$(\mu) \subseteq e^{i\vartheta} U_k \cap e^{i\varphi} U_n$ for some $\vartheta, \varphi \in [0, 2\pi)$. In the case $|1-2w| = 1$, if supp(μ) consists of one point then equality holds.

For $w \in \mathbb{C}$ and $p \in \mathcal{P}$ we define the $(k+1) \times (k+1)$ determinant

$$A_{k,n}(w) = \begin{vmatrix} p_{n+k} & p_{n+k-1} & p_{n+k-2} & \ldots & p_{n+1} & p_n \\ wp_1 & 1 & 0 & \ldots & 0 & 0 \\ wp_2 & wp_1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ wp_{k-1} & wp_{k-2} & wp_{k-3} & \ldots & 1 & 0 \\ wp_k & wp_{k-1} & wp_{k-2} & \ldots & wp_1 & 1 \end{vmatrix}.$$

Livingston [56] defined this for $w = 1$ and proved that $|A_{k,n}(1)| \leq 2$. When no confusion arises we will suppress w and write $A_{k,n}$ for $A_{k,n}(w)$. Here are some examples of initial $A_{k,n}$'s:

$$\begin{aligned} A_{0,n} &= p_n, \qquad A_{1,n} = p_{n+1} - wp_1p_n, \\ A_{2,n} &= p_{n+2} - wp_1p_{n+1} - wp_2p_n + w^2p_1^2p_n. \end{aligned}$$

The following theorem generalizes Livingston's result.

Theorem 1.2 (Complex Determinant Inequality)**.** *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then*

$$|A_{k,n}(w)| \leq 2\max\{1, |1-2w|^k\}$$

for all $k \geq 0$ and $n \geq 1$.

Let μ be the Herglotz measure of p. In the case $|1-2w| < 1$, equality holds if and only if supp$(\mu) \subseteq e^{i\varphi}U_{n+k}$ for some $\varphi \in [0, 2\pi)$ and $p_1 = p_2 = \ldots = p_k = 0$. In the case $|1-2w| \geq 1$, if supp(μ) consists of one point then equality holds.

We observe that in Theorem 1.1 the condition for equality in the case $|1-2w| = 1$ is far from being necessary. To illustrate this consider $w = 1$, $n = 2k$ and a Herglotz measure supported on two arbitrary points λ_1, λ_2 on $\mathbb{T}$ having equal point masses, 1/2 each. Then the coefficients of the corresponding function in $\mathcal{P}$ are $p_j = \lambda_1^j + \lambda_2^j$ and one easily computes

$$|p_{2k} - p_k^2| = |\lambda_1^{2k} + \lambda_2^{2k} - (\lambda_1^k + \lambda_2^k)^2| = 2.$$

The complete characterization of equality when $|1-2w| = 1$ will be given in Theorem 1.6. Here we only mention the special case where $w = 1$ (note that this was not explicitly stated in [55]): It holds that $|p_n - p_k p_{n-k}| = 2$ if and only if either

(i) $p_k = 0$ and $supp(\mu) \subseteq e^{i\varphi} U_n$ for some φ in $[0, 2\pi)$; or

(ii) $p_k \neq 0$,

$$supp(\mu) \subseteq (e^{i\varphi} U_{n-2k} \cap e^{i\vartheta_1} U_k) \cup (e^{i\varphi} U_{n-2k} \cap e^{i\vartheta_2} U_k)$$

for some φ, ϑ_1 and ϑ_2 in $[0, 2\pi)$ and, except for the degenerate case where the support of μ consists of only one point, the total mass of the measure in each of the two sets of the union is equal to $1/2$.

Both Theorems 1.1 and 1.2 have a version for non-normalized functions $p(z) = \sum_{n=0}^{\infty} p_n z^n$ with positive real part. For such a function p, let $p_0 = x + iy, (x > 0)$ and $q(z) = \big(p(z) - iy\big)/x$, which is obviously a function in $\mathcal{P}$. To this q, with coefficients $q_n = p_n/x$, we can apply Theorems 1.1 and 1.2. Then multiply both inequalities by $x/|p_0|$ and set wx/p_0 instead of w. What results is

$$\left| \frac{p_n}{p_0} - w \frac{p_k p_{n-k}}{p_0^2} \right| \leq 2 \frac{\operatorname{Re} p_0}{|p_0|} \max \left\{ 1, \left| 1 - \frac{2w \operatorname{Re} p_0}{p_0} \right| \right\}$$

and

$$|A_{k,n}| \leq 2 \frac{\operatorname{Re} p_0}{|p_0|} \max \left\{ 1, \left| 1 - \frac{2w \operatorname{Re} p_0}{p_0} \right|^k \right\}$$

for the modified $A_{k,n}$, having p_j/p_0 in place of p_j (for all j). Note that for $w = 1$ the two quantities in the maximum are equal and what one gets is Livingston's original results.

An alternative proof for the inequality in Theorem 1.2 under the additional condition $n \geq k + 1$ can be given via the method of Delsarte and Genin [21]. Their approach relies on the observation that $A_{k,n}(1)$ is related to a truncation of the reciprocal of a function in $\mathcal{P}$. With the aid of Herglotz' formula they get a substantially simpler proof of Livingston's result. The proof, which will be presented in Section 1.3, is an adaptation of their arguments to our case of $A_{k,n}(w)$ for any $w \in \mathbb{C}$.

Finally, we turn to a question raised by Goodman [37, p.104] about the sharp bound of $|p_{n+1} - p_n|$ for functions in $\mathcal{P}$ with prescribed p_1. Using extreme point theory, Brown [12] proved the following theorem, for which we will provide a simpler proof.

Theorem 1.3. *Let* $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ *be in* $\mathcal{P}$, $m, n \in \mathbb{N}$ *and* $\nu \in \mathbb{R}$. *Then*

$$|e^{i\nu} p_{n+m} - p_n| \le 2\sqrt{2 - \mathrm{Re}\,(e^{i\nu} p_m)}.$$

The result is sharp.

1.2 A reformulation of the complex Livingston inequality

For a fixed $p \in \mathcal{P}$ the infinitely many conditions of inequality (1.1) in Theorem 1.1 can be shown to be equivalent to just a single condition of different type. We will employ the following simple but very useful lemma for complex numbers.

Lemma 1.4 ([32]). *Let* a, $b \in \mathbb{C}$ *be arbitrary and* C, $M > 0$. *Then*

$$|a + \lambda b| \le M \max\{C, |\lambda|\} \quad \textit{for all} \quad \lambda \in \mathbb{C} \tag{1.2}$$

if and only if

$$|a| + |b|C \le MC\,. \tag{1.3}$$

Assuming that $a, b \neq 0$, *equality holds in* (1.2) *for some* $\lambda \neq 0$ *if and only if it holds in* (1.3) *and* $\lambda = C \exp\big(i \arg(a/b)\big)$.

Proof. Suppose first that (1.2) holds. If any of the numbers a or b is zero then by choosing $|\lambda| = C$ we get (1.3). If $a, b \neq 0$ then we can choose λ with $|\lambda| = C$ and $\arg \lambda = \arg a - \arg b$ to get $|a| + |b|C \le MC$.

Conversely, assuming that $|a| + |b|C \le MC$, the triangle inequality yields

$$\begin{aligned} |a + \lambda b| &\le |a| + \left|\frac{\lambda}{C}\right| |b|C \\ &\le \max\left\{1, \left|\frac{\lambda}{C}\right|\right\} |a| + \max\left\{1, \left|\frac{\lambda}{C}\right|\right\} |b|C \\ &\le MC \max\left\{1, \left|\frac{\lambda}{C}\right|\right\} = M \max\{C, |\lambda|\}\,. \end{aligned}$$

By inspecting the above chain of inequalities we see that equality in (1.2) is possible only when $\lambda = 0$ or $|\lambda| = C$ and $\arg a = \arg(\lambda b)$. We now observe that for the latter value of λ the inequalities (1.2) and (1.3) coincide.

$\square$

We can now deduce an inequality that can be seen as a generalization of the well-known estimate

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_1|^2}{2},$$

which can be found in [62, p. 166] and can also be derived from the classical Schwarz-Pick lemma.

Proposition 1.5. *For any $p \in \mathcal{P}$ it holds that*

$$\left| p_n - \frac{p_k p_{n-k}}{2} \right| \le 2 - \frac{|p_k p_{n-k}|}{2}$$

for all $1 \le k \le n-1$. The inequality is sharp.

Proof. Rewriting inequality (1.1) from Theorem 1.1 in the form

$$|2p_n - p_k p_{n-k} + (1-2w) p_k p_{n-k}| \le 4 \max\{1, |1-2w|\},$$

the statement follows by Lemma 1.4. $\square$

We note that the inequality stated in Proposition 1.5 also appeared (with a different proof) in Campschroer's thesis [15, §1.4].

1.3 Proofs of the results

Proof of Theorem 1.1 (Complex Livingston Inequality). First we note that $|1-2w| \leq 1$ if and only if $|w|^2 \leq \operatorname{Re} w$. We compute

$$\begin{aligned}
|p_n - wp_kp_{n-k}| &= \left| 2\int_{\mathbb{T}} \lambda^n d\mu(\lambda) - 2wp_k \int_{\mathbb{T}} \lambda^{n-k} d\mu(\lambda) \right| \\
&\leq 2\int_{\mathbb{T}} |\lambda^n - wp_k\lambda^{n-k}| d\mu(\lambda) \\
&\leq \ 2\left(\int_{\mathbb{T}} |\lambda^k - wp_k|^2 d\mu(\lambda)\right)^{1/2} \\
&= 2\left(\int_{\mathbb{T}} 1 - 2\operatorname{Re}(wp_k\lambda^{-k}) + |wp_k|^2 d\mu(\lambda)\right)^{1/2} \\
&= 2\Big(1 - 2\operatorname{Re}(wp_k\overline{p_k}/2) + |wp_k|^2\Big)^{1/2} \\
&= 2\Big(1 + (|w|^2 - \operatorname{Re} w)|p_k|^2\Big)^{1/2} \\
&\leq 2\max\{1, |1-2w|\}.
\end{aligned}$$

Here we used the triangle and Cauchy-Schwarz inequalities. At the last step, in the case $|1-2w| > 1$, we made use of Theorem 0.1.

Now suppose that equality holds. If $|1-2w| < 1$ then equality in the last of the above inequalities yields $p_k = 0$. Hence the second term in $p_n - wp_kp_{n-k}$ vanishes and we have $|p_n| = 2$. By Theorem 0.1, $supp(\mu) \subseteq e^{i\varphi}U_n$ for some $\varphi \in [0, 2\pi)$.

In the case $|1-2w| > 1$, the last inequality yields $|p_k| = 2$. Hence $supp(\mu) \subseteq e^{i\vartheta}U_k$ for some $\vartheta \in [0, 2\pi)$. Now $p_k = 2e^{ik\vartheta}$ and

$$p_{n-k} = 2\int_{\mathbb{T}} \lambda^{n-k} d\mu(\lambda) = 2e^{-ik\vartheta}\int_{\mathbb{T}} \lambda^n d\mu(\lambda) = e^{-ik\vartheta}p_n.$$

Hence $2|1-2w| = |p_n - 2wp_n|$, which implies that $|p_n| = 2$. Again by Theorem 0.1 we have $supp(\mu) \subseteq e^{i\varphi}U_n$ for some $\varphi \in [0, 2\pi)$ and thus $supp(\mu)$ must form a subset of the intersection $e^{i\vartheta}U_k \cap e^{i\varphi}U_n$.

It is elementary to check that in all three cases the conditions are sufficient for equality. $\square$

Proof of Theorem 1.2 (Complex Determinant Inequality). Let $n \geq 1$ and $w \in \mathbb{C}$ be fixed. The case $k = 0$ follows from Theorem 0.1. For $k \geq 1$ we define

$$Q_{k,n}(\lambda) = \begin{vmatrix} \lambda^{n+k-1} & p_{n+k-1} & p_{n+k-2} & \dots & p_n \\ w & 1 & 0 & \dots & 0 \\ w\lambda & wp_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w\lambda^{k-1} & wp_{k-1} & wp_{k-2} & \dots & 1 \end{vmatrix}.$$

Expanding $A_{k,n}$ along the first column, using the Herglotz formula and the linearity of the integral, and finally putting the determinant back together, we get $A_{k,n} = 2\int_{\mathbb{T}} \lambda Q_{k,n}(\lambda) d\mu(\lambda)$.

We will now show by induction that

$$\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \leq \max\{1, |1-2w|^{2k}\} \tag{1.4}$$

for all $k \geq 1$. Then the desired inequality will follow since

$$\begin{aligned} |A_{k,n}| &\leq 2\int_{\mathbb{T}} |Q_{k,n}(\lambda)| d\mu(\lambda) \\ &\leq 2\left(\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda)\right)^{1/2} \\ &\leq 2\max\{1, |1-2w|^k\}, \end{aligned} \tag{1.5}$$

by the triangle and Cauchy-Schwarz inequalities.

We first prove (1.4) for $k = 1$. (Recall that $|1-2w| < 1$ if and only if $|w|^2 < \operatorname{Re} w$.)

$$\begin{aligned} \int_{\mathbb{T}} |Q_{1,n}(\lambda)|^2 d\mu(\lambda) = &\int_{\mathbb{T}} 1 + |wp_n|^2 - 2\mathrm{Re}\,(wp_n\lambda^{-n}) d\mu(\lambda) \\ =&1 + (|w|^2 - \operatorname{Re} w)|p_n|^2 \\ \leq&\max\{1, |1-2w|^2\}. \end{aligned} \tag{1.6}$$

Next, let us assume that (1.4) holds for k and let us prove it for $k+1$ instead of k. Expanding $Q_{j,n}(\lambda)$ along the second row it is not difficult to see that

$$\begin{aligned} Q_{j,n}(\lambda) = & \begin{vmatrix} \lambda^{n+j-1} & p_{n+j-2} & \dots & p_n \\ w\lambda & 1 & \dots & 0 \\ w\lambda^2 & wp_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w\lambda^{j-1} & wp_{j-2} & \dots & 1 \end{vmatrix} - w \begin{vmatrix} p_{n+j-1} & p_{n+j-2} & \dots & p_n \\ wp_1 & 1 & \dots & 0 \\ wp_2 & wp_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ wp_{j-1} & wp_{j-2} & \dots & 1 \end{vmatrix} \\ = & \lambda Q_{j-1,n}(\lambda) - wA_{j-1,n}. \end{aligned}$$

For $j = k+1$ we have $Q_{k+1,n}(\lambda) = \lambda Q_{k,n}(\lambda) - wA_{k,n}$. Hence

$$\begin{aligned} & \int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) = \\ & \quad = \int_{\mathbb{T}} \left[|Q_{k,n}(\lambda)|^2 - 2\mathrm{Re}\left(w\overline{\lambda Q_{k,n}(\lambda)} A_{k,n} \right) \right] d\mu(\lambda) + |w|^2 |A_{k,n}|^2 \\ & \quad = \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) + (|w|^2 - \mathrm{Re}\, w)|A_{k,n}|^2. \end{aligned} \tag{1.7}$$

We distinguish two cases. If $|1-2w| < 1$ then (1.7) and (1.4) show that

$$\int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) \le \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \le 1. \tag{1.8}$$

For the case $|1-2w| \ge 1$, we make a further use of the Cauchy-Schwarz inequality to obtain $|A_{k,n}|^2 \le 4\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda)$. Now by (1.7) and (1.4) we get that

$$\begin{aligned} \int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) &\le (1 + 4|w|^2 - 4\mathrm{Re}\, w) \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \\ &\le |1-2w|^2 |1-2w|^{2k} \\ &= |1-2w|^{2k+2}. \end{aligned}$$

Hence (1.4) has been proved for all $k \ge 1$.

We now turn to the case of equality. Suppose that $|1-2w| < 1$ and $|A_{k,n}| = 2$. Then inequalities (1.5) become equalities and in particular $\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) = 1$. The inductive step (1.8) shows that we must have $\int_{\mathbb{T}} |Q_{j,n}(\lambda)|^2 d\mu(\lambda) = 1$ for all $j = 1, 2, \dots, k$. This is true in particular for $j = 1$, which by (1.6) implies that $p_n = 0$. This in turn is easily seen to imply that $A_{k,n} = A_{k-1,n+1}$. Hence we may repeat the above argument to get that $\int_{\mathbb{T}} |Q_{j,n+1}(\lambda)|^2 d\mu(\lambda) = 1$ for all $j = 1, 2, \ldots, k-1$. Again from $j = 1$ we get by (1.6) that $p_{n+1} = 0$. We repeat this argument until we get $p_n = p_{n+1} = \ldots = p_{n+k-1} = 0$. Now $A_{k,n} = A_{0,n+k} = p_{n+k}$ is a number of modulus 2 and therefore Theorem 0.1 yields $supp(\mu) \subseteq e^{i\varphi} U_{n+k}$ for some $\varphi \in [0, 2\pi)$. Finally, for all $j = 1, 2, \ldots, k$, we have

$$p_j = 2\int_{\mathbb{T}} \lambda^j d\mu(\lambda) = 2e^{i(n+k)\varphi} \int_{\mathbb{T}} \lambda^{j-n-k} d\mu(\lambda) = e^{i(n+k)\varphi}\overline{p_{n+k-j}} = 0.$$

In both cases the sufficiency for equality is easy to verify. □

Alternative proof of Theorem 1.2 (case $n \geq k+1$). Let $w \in \mathbb{C}$ be fixed. The case $k = 0$ follows from Theorem 0.1. Let $k \geq 1$ and consider the perturbation

$$p^*(z) = 1 + w(p_1 z + \ldots + p_k z^k) + p_{k+1} z^{k+1} + \ldots$$

Let $Q_k(z) = 1 + q_1 z + \ldots + q_k z^k$ be the k^{th} partial sum of $(p^*)^{-1}$, the reciprocal of p^*. We define $v_k(z) = \sum_{m=0}^{\infty} v_{k,m} z^m$, analytic at the origin, via the identity

$$Q_k(z) p^*(z) = 1 + 2z^{k+1} v_k(z).$$

Computing the coefficient of z^{k+m+1}, for $m \geq k$, we get that

$$2v_{k,m} = \sum_{j=0}^{k} q_j \, p_{k+m+1-j}. \tag{1.9}$$

Note that for $k_1 \neq k_2$ the coefficients q_j coincide for $1 \leq j \leq \min\{k_1, k_2\}$, hence formula (1.9) readily implies that

$$2v_{k,m} = q_k p_{m+1} + 2v_{k-1,m+1}. \tag{1.10}$$

We now proceed by induction on $k \geq 1$ to prove that

$$2v_{k,m} = A_{k,m+1}(w) \quad \text{for all} \quad m \geq k. \tag{1.11}$$

For $k = 1$ it is easy to verify that $2v_{1,m} = p_{m+2} - wp_1 p_{m+1} = A_{1,m+1}$ for all $m \geq 1$.

Next we suppose that (1.11) holds for some k. We shall prove it for $k+1$ instead of k. Expanding with respect to the last column we see that

$$\begin{aligned} A_{k+1,m+1} &= A_{k,m+2} + p_{m+1}(-1)^{k+1} \begin{vmatrix} wp_1 & 1 & \dots & 0 \\ wp_2 & wp_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ wp_k & wp_{k-1} & \dots & 1 \\ wp_{k+1} & wp_k & \dots & wp_1 \end{vmatrix} \\ &= A_{k,m+2} + p_{m+1}q_{k+1}, \end{aligned}$$

where we made use of Wronski's formula [43, p.17] for the coefficients of the reciprocal of a power series. Therefore by (1.10) we get that

$$A_{k+1,m+1} = 2v_{k,m+1} + p_{m+1}q_{k+1} = 2v_{k+1,m}$$

for $m \geq k+1$. Thus (1.11) has been proved. We set $m = n-1$ and write $A_{k,n}(w) = 2v_{k,n-1}$ for $n \geq k+1$.

We proceed as in [21] using the Herglotz formula in (1.9) and the Cauchy-Schwarz inequality to get

$$|v_{k,n-1}|^2 = \left| \int_{\mathbb{T}} \lambda^{k+n} Q_k(\overline{\lambda}) d\mu(\lambda) \right|^2 \leq \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda).$$

Now, we show that

$$\int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda) \leq \max\{1, |1-2w|^{2k}\} \tag{1.12}$$

by induction on $k \geq 1$.

For $k = 1$ we compute

$$\int_{\mathbb{T}} |Q_1(\overline{\lambda})|^2 d\mu(\lambda) = 1 + |p_1|^2(|w|^2 - \operatorname{Re} w) \leq \max\{1, |1-2w|^2\}.$$

Now we suppose that (1.12) is true for k. We shall prove it for $k+1$ instead of k. We compute

$$\begin{aligned} \int_{\mathbb{T}} |Q_{k+1}(\overline{\lambda})|^2 d\mu(\lambda) &= \int_{\mathbb{T}} \sum_{j,m=0}^{k+1} q_j \overline{q_m} \lambda^{m-j} d\mu(\lambda) \\ &= \sum_{j=0}^{k+1} |q_j|^2 + \operatorname{Re}\left(\sum_{j<m} q_j \overline{q_m} p_{m-j} \right), \end{aligned}$$

where $j = 0, 1, \ldots, k$ and $m = 1, 2, \ldots, k+1$ at the last summation. Therefore

$$\int_{\mathbb{T}} |Q_{k+1}(\overline{\lambda})|^2 d\mu(\lambda) = \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda) + |q_{k+1}|^2 + \mathrm{Re}\left(\overline{q_{k+1}} \sum_{j=0}^{k} q_j p_{k+1-j}\right)$$

$$= \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda) + (|w|^2 - \mathrm{Re}\, w) \left|\sum_{j=0}^{k} q_j p_{k+1-j}\right|^2,$$

since $q_{k+1} + w \sum_{j=0}^{k} q_j p_{k+1-j} = 0$ by the definition of Q_{k+1}. If $|1 - 2w| < 1$ then

$$\int_{\mathbb{T}} |Q_{k+1}(\overline{\lambda})|^2 d\mu(\lambda) \leq \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda) \leq 1$$

and we are done. If $|1-2w| \geq 1$ then we make a further use of the Herglotz formula to get

$$\left|\sum_{j=0}^{k} q_j p_{k+1-j}\right|^2 = \left|2\int_{\mathbb{T}} \lambda^{k+1} Q_k(\overline{\lambda}) d\mu(\lambda)\right|^2 \leq 4 \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda).$$

Hence

$$\int_{\mathbb{T}} |Q_{k+1}(\overline{\lambda})|^2 d\mu(\lambda) \leq (1 + 4|w|^2 - 4\mathrm{Re}\, w) \int_{\mathbb{T}} |Q_k(\overline{\lambda})|^2 d\mu(\lambda) \leq |1 - 2w|^{2k+2}$$

and (1.12) has been established.

It is not clear how one can make the above argument work when $n \leq k$. $\square$

Proof of Theorem 1.3 (Brown's Theorem). The proof relies on a further generalization of Theorem 1.1. Let $w \in \mathbb{C}$ and compute

$$\begin{aligned}
|p_{n+m} - wp_n| &\leq 2\int_{\mathbb{T}} |\lambda^m - w| d\mu(\lambda) \\
&\leq 2\left(\int_{\mathbb{T}} |\lambda^m - w|^2 d\mu(\lambda)\right)^{1/2} \\
&= 2\Big(1 + |w|^2 - \mathrm{Re}\,(\overline{w} p_m)\Big)^{1/2}.
\end{aligned}$$

Choosing $w = e^{-i\nu}$ we obtain the desired inequality. Equality evidently holds for the half-plane function $\frac{1+z}{1-z}$. $\square$

1.4 The case of equality in the complex Livingston inequality

We now consider the case of equality for Theorem 1.1 when $|1-2w| = 1$. Since our result is more general than Livingston's, the analysis of the conditions for equality and their proofs are more lengthy.

Theorem 1.6. *Let $p \in \mathcal{P}$, μ be its representing Herglotz measure, $1 \leq k \leq n-1$ and $w = (1+e^{i\vartheta})/2$ with $|\vartheta| < \pi$. Then $p_n - wp_kp_{n-k} = 2e^{ic}$ for some c in $[0, 2\pi)$ if and only if either*

(i) *$p_k = 0$ and $supp(\mu) \subseteq e^{ic/n}U_n$; or*

(ii) *$p_k \neq 0$,*

$$supp(\mu) \subseteq (e^{i\frac{\psi}{n-2k}}U_{n-2k} \cap e^{i\left(\frac{\varphi}{k}+\frac{c-\psi}{2k}\right)}U_k) \cup (e^{i\frac{\psi}{n-2k}}U_{n-2k} \cap e^{i\left(\frac{\pi-\varphi}{k}+\frac{c-\psi}{2k}\right)}U_k) \tag{1.13}$$

for some ψ in $[0, 2\pi)$ and $|\varphi| \leq \pi/2$ and, except for the degenerate case where the support of μ consists of only one point, the total mass of the measure in each of the two sets of the union is (respectively) equal to

$$\frac{1}{2}\left(1+\frac{\sin\vartheta}{1+\cos\vartheta}\tan\varphi\right) \quad \text{and} \quad \frac{1}{2}\left(1-\frac{\sin\vartheta}{1+\cos\vartheta}\tan\varphi\right).$$

Proof. We observe that without loss of generality we may assume that $2k \leq n$, since otherwise, we may set $m = n-k$ and see that the functional $p_n - wp_kp_{n-k}$ remains unchanged while the new pair of integers (m, n) satisfies $2m < n$. Therefore the second condition makes sense.

We will prove the necessity of the two conditions, since the sufficiency is elementary, although laborious in the case (ii).

We assume that $c = 0$. Having proved the assertion in this case we apply it to the rotated function $p(e^{-ic/n}z)$ in order to obtain the general result.

Reviewing the equalities in the proof of Theorem 1.1 we see that

$$\lambda^n - wp_k\lambda^{n-k} = 1, \qquad \lambda \in supp(\mu), \tag{1.14}$$

since equality in the triangle inequality yields constant argument and equality in the Cauchy-Schwarz inequality yields constant modulus. Formula (1.14) is equivalent to $\lambda^k - wp_k = \lambda^{k-n}$, which we integrate with respect to μ in order to get

$$p_{n-k} = (1 - 2\overline{w})\overline{p_k} = -e^{-i\vartheta}\overline{p_k}. \tag{1.15}$$

It is now evident that if one of the coefficients p_k, p_{n-k} is zero, then both of them are zero. If $p_k = 0$, case (i) clearly follows from Theorem 0.1, but it can also be seen from (1.14) which becomes $\lambda^n = 1$.

Suppose that $p_k \neq 0$. In order to prove condition (ii) we begin with the additional assumption that $n = 2k$. Equation (1.14) is then equivalent to $\lambda^k - \lambda^{-k} = wp_k$. From this we deduce that $\operatorname{Im}\lambda^k$ is constant on $supp(\mu)$ and that $\operatorname{Re}(wp_k) = 0$. The former implies that for some $\zeta = e^{i\varphi}$ (we may assume that $|\varphi| \leq \pi/2$), the support of μ consists of the k-th roots of ζ and $-\overline{\zeta}$, having point masses, say, m_j and m_j^*, respectively, $1 \leq j \leq k$. In other words

$$supp(\mu) \subseteq e^{i\frac{\varphi}{k}} U_k \cup e^{i\frac{\pi-\varphi}{k}} U_k, \tag{1.16}$$

with total mass in each of the two sets of the union $M = \sum_{j=1}^k m_j$ and $M^* = \sum_{j=1}^k m_j^*$, respectively. The fact that μ is a probability measure means that $M + M^* = 1$. Next, we easily see that $p_k = 2\int_{\mathbb{T}} \lambda^k d\mu(\lambda) = 2(\zeta M - \overline{\zeta} M^*) = 2\left((\zeta + \overline{\zeta})M - \overline{\zeta}\right)$. Hence

$$\begin{aligned} 0 = \operatorname{Re}(w\, p_k) &= \operatorname{Re}\left[(1 + e^{i\vartheta})\left((\zeta + \overline{\zeta})M - \overline{\zeta}\right)\right] \\ &= (1 + \cos\vartheta)\cos\varphi\,(2M - 1) - \sin\varphi\sin\vartheta. \end{aligned} \tag{1.17}$$

If $|\varphi| = \pi/2$, *i.e.* if ζ is either i or $-i$, then ζ and $-\overline{\zeta}$ coincide and therefore we may choose to divide the total mass of μ into two parts in any possible way, and in particular as asserted in (ii). Otherwise, if $|\varphi| < \pi/2$, equation (1.17) implies

$$M = \frac{1}{2}\left(1 + \frac{\sin\vartheta}{1 + \cos\vartheta}\tan\varphi\right).$$

Hence, to see that (1.13) has been proved, recall that we regard U_0 as $\mathbb{T}$ and therefore, since $n = 2k$, we may choose ψ freely. The choice $\psi = 0$ completes the proof of (1.13) in case $n = 2k$.

For the remaining case $n > 2k$ in the case (ii), we repeat the arguments used to prove (1.14) to get

$$\lambda^n - wp_{n-k}\lambda^k = 1, \qquad \lambda \in supp(\mu). \tag{1.18}$$

A combination of (1.14) and (1.18) shows that $p_k\lambda^{n-k} = p_{n-k}\lambda^k$. Hence, by (1.15),

$$\lambda^{n-2k} = -e^{-i\vartheta}\overline{p_k}/p_k, \qquad \lambda \in supp(\mu).$$

This yields

$$supp(\mu) \subseteq e^{i\frac{t}{n-2k}} U_{n-2k} \tag{1.19}$$

for some $t \in [0, 2\pi)$. Hence $p_n = e^{it}p_{2k}$, $p_{n-k} = e^{it}p_k$ and $2 = p_n - wp_kp_{n-k}$ $= e^{it}(p_{2k} - wp_k^2)$. It follows that the function $p(e^{it/2k}z)$ must satisfy condition (1.16) and, therefore, $p(z)$ satisfies the corresponding rotation of (1.16). Together with (1.19) this is

$$supp(\mu) \subseteq (e^{i\frac{t}{n-2k}} U_{n-2k} \cap e^{i\left(\frac{\varphi}{k} - \frac{t}{2k}\right)} U_k) \cup (e^{i\frac{t}{n-2k}} U_{n-2k} \cap e^{i\left(\frac{\pi-\varphi}{k} - \frac{t}{2k}\right)} U_k),$$

which is (1.13) in case $c = 0$. If $c \neq 0$ then a further rotation by $e^{ic/n}$ and the substitution $\psi = t + c(1 - 2k/n)$ yield (1.13). □

1.5 Application to the self-maps of $\mathbb{D}$

There is a close connection between the class $\mathcal{P}$ and self-maps of $\mathbb{D}$ via conformal maps of $\mathbb{D}$ to the right half-plane, namely, $p = \frac{1+\varphi}{1-\varphi}$ is in $\mathcal{P}$ for a function φ analytic in $\mathbb{D}$ if and only if $\varphi : \mathbb{D} \to \mathbb{D}$ and $\varphi(0) = 0$. Writing $\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$ we may relate the first few coefficients of the two functions by

$$\begin{aligned} p_1 =& 2a_1, \qquad p_2 = 2(a_2 + a_1^2), \qquad p_3 = 2(a_3 + 2a_1a_2 + a_1^3), \\ p_4 =& 2(a_4 + 2a_1a_3 + a_2^2 + 3a_1^2a_2 + a_1^4). \end{aligned}$$

For functions φ of this form, the Schwarz lemma states that $|a_1| \leq 1$ while the Schwarz-Pick lemma says that $|a_2| \leq 1 - |a_1|^2$. One then easily computes

$$|a_2 + \lambda a_1^2| \leq |a_2| + |\lambda||a_1|^2 \leq 1 + (|\lambda| - 1)|a_1|^2 \leq \max\{1, |\lambda|\}.$$

(See [47] for this calculation and an application of it.) The same inequality can be obtained from our Theorem 1.1 with $\lambda = 1 - 2w$ and $n = k + 1 = 2$.

For higher order coefficients one has F.W. Wiener's generalization of the Schwarz-Pick lemma $|a_n| \le 1 - |a_1|^2$ (see [7] or problem 9 on p.172 of [60]). However, even if we use this inequality, it does not seem easy to get the following corollary in a different way, without applying our Theorems 1.1 and 1.2.

Corollary. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, $\varphi(0) = 0$ and $\lambda \in \mathbb{C}$ then*

$$|a_3 + (1+\lambda)a_1a_2 + \lambda a_1^3| \le \max\{1, |\lambda|\} \tag{1.20}$$

$$|a_3 + 2\lambda a_1a_2 + \lambda^2 a_1^3| \le \max\{1, |\lambda|^2\} \tag{1.21}$$

$$|a_3 + a_1a_2| + |a_1a_2 + a_1^3| \le 1 \tag{1.22}$$

and

$$|a_4 + (1+\lambda)a_1a_3 + a_2^2 + (1+2\lambda)a_1^2a_2 + \lambda a_1^4| \le \max\{1, |\lambda|\} \tag{1.23}$$

$$|a_4 + 2a_1a_3 + \lambda a_2^2 + (1+2\lambda)a_1^2a_2 + \lambda a_1^4| \le \max\{1, |\lambda|\} \tag{1.24}$$

$$|a_4 + (1+\lambda)a_1a_3 + \lambda a_2^2 + \lambda(2+\lambda)a_1^2a_2 + \lambda^2 a_1^4| \le \max\{1, |\lambda|^2\} \tag{1.25}$$

$$|a_4 + 2\lambda a_1a_3 + \lambda a_2^2 + 3\lambda^2 a_1^2a_2 + \lambda^3 a_1^4| \le \max\{1, |\lambda|^3\} \tag{1.26}$$

$$|a_4 + 2a_1a_3 + a_1^2a_2| + |a_2^2 + 2a_1^2a_2 + a_1^4| \le 1 \tag{1.27}$$

$$|a_4 + a_1a_3 + a_2^2 + a_1^2a_2| + |a_1a_3 + 2a_1^2a_2 + a_1^4| \le 1. \tag{1.28}$$

Proof. Set $\lambda = 1 - 2w$ and apply Theorem 1.1 with $n = k + 2 = 3$ to get (1.20), with $n = k + 3 = 4$ to get (1.23) and with $n = k + 2 = 4$ to get (1.24). Apply Theorem 1.2 with $k = n+1 = 2$ to get (1.21), with $k = n = 2$ to get (1.25) and with $k = n + 2 = 3$ to get (1.26).

Inequalities (1.22), (1.27) and (1.28), which do not involve the parameter λ, follow from Proposition 1.5 upon setting $n = k + 2 = 3$, $n = k + 2 = 4$ and $n = k + 3 = 4$, respectively. $\square$

Chapter 2

Zalcman's conjecture

2.1 Formulation and earlier results

The functions in the class S, *i.e.*, the normalized univalent functions:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \qquad z \in \mathbb{D},$$

satisfy the estimate $|a_2^2 - a_3| \leq 1$, as we have seen in Corollary 0.4. *Zalcman's conjecture* states that every f in S satisfies the more general sharp inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \qquad n \geq 2.$$

The importance of the conjecture stems from the fact that it implies the Bieberbach conjecture $|a_n| \leq n$, which was solved by de Branges in 1984. This was observed by Zalcman himself in the early 1970s (unpublished); Brown and Tsao [13] gave a slick short proof. They also proved the conjecture for the typically real and starlike functions [13]. Ma [57] did it for the closed convex hull of close-to-convex functions while Krushkal [49, 50] proved it in the general case for small values of n. More general versions of Zalcman's conjecture have also been considered [13, 58, 52, 53] for the functionals such as $\Phi(f) = \lambda\, a_n^2 - a_{2n-1}$ and $\Phi(f) = \lambda a_m a_n - a_{m+n-1}$ for certain positive values of λ. These functionals can be seen as generalizations of the Fekete-Szegö functional $\lambda a_2^2 - a_3$ (mentioned in Section 0.4), but they are also important because they appear frequently in the coefficient formulas for the inversion transformation in the theory of univalent functions [23, Ch. 2, p. 28].

2.2 Sharp estimates for some special classes

In this section we will obtain various estimates on the generalized Zalcman functional $\Phi(f) = \lambda a_m a_n - a_{m+n-1}$ with complex values λ. We do this for four different classes of functions which are either subclasses of S or closed convex hulls of important subclasses of S (which also contain non-univalent functions). All estimates are sharp and each one of them is also formulated in an equivalent way. Definitions, examples and basic properties of these classes were given in Section 0.3.

The Hurwitz class. Recall that the Hurwitz class $\mathcal{H}$ consists of functions $f \in H(\mathbb{D})$ which are normalized and have the property that

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$

For the functions in this class we obtain a much smaller bound on the Zalcman functional than for the entire class S. We stress the difference between items (a) and (b) of the theorem below: the estimates on the functional $\Phi(f) = \lambda a_m a_n - a_{m+n-1}$ differ in an essential way in the cases $m = n$ and $m \neq n$, with the presence of an extra factor of four in the denominator in the latter case.

Theorem 2.1. *(a) If $f \in \mathcal{H}$ and $n \geq 2$ then the following inequality holds for the coefficients of f:*

$$n^2|a_n^2| + (2n-1)|a_{2n-1}| \leq 1\,. \tag{2.1}$$

This single inequality is equivalent to

$$|\lambda\, a_n^2 - a_{2n-1}| \leq \max\left\{\frac{|\lambda|}{n^2}, \frac{1}{2n-1}\right\} \quad \textit{for all } \lambda \in \mathbb{C}\,. \tag{2.2}$$

Equality holds in (2.2) *if and only if*

$$f(z) = \begin{cases} z + \frac{\alpha}{2n-1}\, z^{2n-1}, & \textit{for } |\lambda| \leq \frac{n^2}{2n-1}, \\ z + \frac{\alpha}{n}\, z^n, & \textit{for } |\lambda| \geq \frac{n^2}{2n-1}, \end{cases}$$

where α is a complex number of modulus one. Equality holds in (2.1) *if and only if f is any of the above two functions.*

(b) If $f \in \mathcal{H}$, then for any two distinct values m, $n \geq 2$ we have

$$4mn|a_m a_n| + (m+n-1)|a_{m+n-1}| \leq 1\,. \tag{2.3}$$

The last inequality is equivalent to

$$|\lambda\, a_m a_n - a_{m+n-1}| \leq \max\left\{\frac{|\lambda|}{4mn}, \frac{1}{m+n-1}\right\} \quad \textit{for all } \lambda \in \mathbb{C}\,. \tag{2.4}$$

Equality holds in(2.4) *if and only if*

$$f(z) = \begin{cases} z + \frac{\alpha}{m+n-1}\, z^{m+n-1}, & \textit{for } |\lambda| \leq \frac{4mn}{m+n-1}, \\ z + \frac{\alpha}{2m}\, z^m + \frac{\beta}{2n}\, z^n, & \textit{for } |\lambda| \geq \frac{4mn}{m+n-1}, \end{cases}$$

where α and β are complex numbers such that $|\alpha| = |\beta| = 1$. Equality holds in (2.3) *if and only if f is any of the above two functions.*

Proof. (a) By the definition of $\mathcal{H}$ we have that $n|a_n| \leq 1$ and therefore

$$n^2|a_n|^2 + (2n-1)|a_{2n-1}| \;\leq\; n|a_n| + (2n-1)|a_{2n-1}| \;\leq\; 1.$$

Taking

$$M = \frac{1}{n^2}\,, \quad C = \frac{n^2}{2n-1}$$

in Lemma 1.4, the above inequality is equivalent to (2.2). Obviously, equality is only possible when $n|a_n| = 1$ or $n|a_n| = 0$. The first case implies that $a_{2n-1} = 0$ and all remaining coefficients are zero. The second yields that $(2n-1)|a_{2n-1}| = 1$ and all remaining coefficients are zero, which easily leads to the desired conclusion.

(b) The proof is slightly more involved in the case $m \neq n$. Set $x = m|a_m|$ and $y = n|a_n|$. Clearly $x, y \geq 0$ and by the definition of $\mathcal{H}$ they satisfy $x + y \leq 1$. This and $(x-y)^2 \geqq 0$ imply

$$4xy \leq (x+y)^2 \leq\; x + y\,.$$

It follows readily from the definition of $\mathcal{H}$ that

$$4mn|a_m a_n| + (m+n-1)|a_{m+n-1}| \leq 1\,.$$

Using Lemma 1.4 with

$$M = \frac{1}{4mn}, \quad C = \frac{4mn}{m+n-1}$$

we see that this is equivalent to (2.4).

If equality holds in (b) then we have that either $m|a_m| = n|a_n| = 0$ or $m|a_m| = n|a_n| = 1/2$, which again easily leads to the claim on extremal functions. $\square$

The Noshiro-Warschawski class. We recall the definition of the Noshiro-Warschawski class

$$\mathcal{R} = \{f \in \mathcal{H}(\mathbb{D}) : \operatorname{Re} f'(z) > 0,\ f(0) = f'(0) - 1 = 0\}.$$

Since $\mathcal{H} \subset \mathcal{R}$, it should not be too surprising to have larger upper bounds for the generalized Zalcman functional among the functions in $\mathcal{R}$ than for those in $\mathcal{H}$. This is indeed the case, as our next result shows.

Theorem 2.2. *Let $f \in \mathcal{R}$ and $m, n \geq 2$. Then the following inequality holds for the coefficients of f:*

$$\left|\frac{mn}{2(m+n-1)} a_m a_n - a_{m+n-1}\right| + \frac{mn|a_m a_n|}{2(m+n-1)} \leq \frac{2}{m+n-1}\,.$$

This is equivalent to

$$|\lambda a_m a_n - a_{m+n-1}| \leq \frac{2}{m+n-1} \max\left\{1, \left|1 - 2\lambda \frac{m+n-1}{mn}\right|\right\} \quad \text{for all} \quad \lambda \in \mathbb{C}\,.$$

Equality holds in both inequalities for the function

$$f(z) = 2 \log \frac{1}{1-z} - z \tag{2.5}$$

when $\left|1 - 2\lambda \frac{m+n-1}{mn}\right| \geq 1$ and for

$$f(z) = \int_{[0,z]} \frac{1 + \zeta^{m+n-2}}{1 - \zeta^{m+n-2}} d\zeta$$

(integrating over the segment from 0 to z) when $\left|1 - 2\lambda \frac{m+n-1}{mn}\right| < 1$.

Proof. Let $f \in \mathcal{R}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$. Then $p = f' \in \mathcal{P}$ and, writing $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, the coefficients of f and p are related by $p_{n-1} = n a_n$. The desired inequalities now follow from Theorem 1.1 and Proposition 1.5.

The function given by (2.5) has coefficients $2/n$ and yields equality in the cases indicated. For the remaining case, when $\left|1 - 2\lambda \frac{m+n-1}{mn}\right| < 1$, we find that the function

$$f'(z) = \frac{1 + z^{m+n-2}}{1 - z^{m+n-2}}$$

belongs to $\mathcal{P}$. Setting $f(0) = 0$, it follows that $f \in \mathcal{R}$. Clearly,

$$f(z) = z + \sum_{k=1}^{\infty} \frac{2}{k(m+n-2)+1} z^{k(m+n-2)+1},$$

and it is easily checked that equality is attained for this function. $\square$

We observe that one can write down an explicit formula for the extremal function written above as a primitive function, but there is really no need for this.

The closed convex hull of convex functions. We write C for the class of convex functions in S, $\mathrm{co}(C)$ for the convex hull of C and $\overline{\mathrm{co}}(C)$ for its closure in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

For real parameters λ and in the case when $m = n$, the inequality in the following theorem appeared in our unpublished preprint [31] for $0 \leq \lambda \leq 2$ and in [53] for $\lambda \geq 2$. Here we give a complete answer for all complex λ and all $m, n \geq 2$.

Theorem 2.3. *Let f be in $\overline{\mathrm{co}}(C)$ and $m, n \geq 2$. Then*

$$|a_m a_n - a_{m+n-1}| + |a_m a_n| \leq 1\,.$$

This is equivalent to the following statement:

$$|\lambda a_m a_n - a_{m+n-1}| \leq \max\{1, |1-\lambda|\} \quad \textit{for all} \quad \lambda \in \mathbb{C}\,.$$

Equality holds in both inequalities for the function given by

$$f(z) = \frac{z}{1-z} \tag{2.6}$$

when $|1-\lambda| \geq 1$ *and for*

$$f(z) = \frac{z}{1-z^{m+n-2}}$$

when $|1-\lambda| < 1$.

Proof. In view of formula (9), the function p given by

$$p(z) \;=\; 2\frac{f(z)}{z} - 1 \;=\; 1 + \sum_{n=1}^{\infty} p_n z^n\,, \qquad z \in \mathbb{D},$$

belongs to $\mathcal{P}$ and the coefficients of the functions f and p are related by $p_{n-1} = 2a_n$. Theorem 1.1 yields the desired inequality in λ and the equivalent formulation as a single inequality follows by Proposition 1.5.

The function given by (2.6) clearly yields equality in the cases indicated. For the remaining case, when $|1-\lambda| < 1$, we consider the function

$$p(z) \;=\; \frac{1+z^{m+n-2}}{1-z^{m+n-2}}\,,$$

which belongs to $\mathcal{P}$. Let f be the function in $\overline{\mathrm{co}}(C)$ for which $p(z) = 2f(z)/z - 1$. We see that

$$f(z) \;=\; \frac{z}{1-z^{m+n-2}} \;=\; \sum_{k=0}^{\infty} z^{k(m+n-2)+1},$$

and that equality is attained for this function. □

The closed convex hull of starlike functions. We write S^* for the class of starlike functions in S and $\overline{\mathrm{co}}(S^*)$ for the closure of the convex hull of S^*.

Brown and Tsao [13, Theorem 2] showed that the Zalcman conjecture is true for starlike functions and Ma [58, Theorem 2.3] generalized their result further to show that

$$|\lambda a_m a_n - a_{m+n-1}| \leq \lambda\, mn - m - n + 1$$

whenever $\lambda \in \mathbb{R}$ and $\lambda \geq \lambda_0 = \frac{2(m+n-1)}{mn}$. The following theorem generalizes his result to the case of complex parameters and at the same time answers in the affirmative his question posed in [58] as to whether λ_0 is the smallest positive number for which the above bound remains true.

Theorem 2.4. *Let $f \in \overline{\text{co}}(S^*)$ and $m, n \geq 2$. Then*

$$\left|\frac{a_m a_n}{mn} - \frac{a_{m+n-1}}{m+n-1}\right| + \frac{|a_m a_n|}{mn} \leq 1\,.$$

This statement is equivalent to

$$|\lambda a_m a_n - a_{m+n-1}| \leq (m+n-1)\max\left\{1, \left|1 - \frac{mn}{m+n-1}\lambda\right|\right\} \quad \text{for all } \lambda \in \mathbb{C}\,.$$

In both cases, equality holds for the function given by

$$f(z) = \frac{z}{(1-z)^2} \tag{2.7}$$

when $\left|1 - \frac{mn}{m+n-1}\lambda\right| \geq 1$ *and for*

$$f(z) = \frac{z}{1-z^{m+n-2}} + (m+n-2)\frac{z^{m+n-1}}{(1-z^{m+n-2})^2} \tag{2.8}$$

when $\left|1 - \frac{mn}{m+n-1}\lambda\right| < 1$.

Proof. In view of formula (10) in Section 0.3 we have that

$$f(z) \;=\; \frac{z}{2}\big(1 + p(z) + zp'(z)\big),$$

for some p in the Carathéodory class $\mathcal{P}$. We write $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and deduce that $a_n = \frac{np_{n-1}}{2}$, $n \geq 2$. Now, Theorem 1.1 yields the first and Proposition 1.5 the second of the two inequalities.

We note that the Koebe function (2.7) clearly satisfies the equality in the cases indicated. For the remaining case, when $\left|1 - \frac{mn}{m+n-1}\lambda\right| < 1$, we consider the function

$$p(z) = \frac{1 + z^{m+n-2}}{1 - z^{m+n-2}}\,,$$

which belongs to $\mathcal{P}$. Hence, the function $f = \frac{z}{2}(1 + p + zp')$ belongs to $\overline{\text{co}}(S^*)$ and has the form (2.8). We now compute

$$f(z) = z + \sum_{k=1}^{\infty}\big(k(m+n-2)+1\big)z^{k(m+n-2)+1}\,.$$

Clearly, equality is attained for this function in both inequalities. □

The class $\overline{\text{co}}(S^*)$ is obviously strictly larger than S^* and it turns out that, in the simplest case $\lambda = 1$, the above Theorem 2.4 yields the sharp bound

$$|a_m a_n - a_{m+n-1}| \leq \max\{m+n-1, (m-1)(n-1)\}\,,$$

which is different from $(m-1)(n-1)$ when either $m = 2$ or $m = n = 3$; this is explained in [58]. In particular, when $m = n \in \{2, 3\}$ we have the estimate $|a_n^2 - a_{2n-1}| \leq 2n - 1$. In this case, $2n - 1 > (n-1)^2$, the general estimate in the Zalcman conjecture (also confirmed by Brown and Tsao for starlike functions). However, there is no contradiction since the class $\overline{\text{co}}(S^*)$ also contains non-univalent functions.

Convex and starlike functions of order α. Recall that $C(\alpha)$ denotes the class of convex functions of order α and $S^*(\alpha)$ the class of starlike functions of order α, respectively defined by

$$\text{Re}\left(1 + \frac{zf''}{f'}\right) > \alpha \qquad \text{and} \qquad \text{Re}\left(\frac{zf'}{f}\right) > \alpha$$

for normalized $f \in H(\mathbb{D})$. Our considerations include any $\alpha < 1$. We have seen that the typical function in $C(\alpha)$ is

$$f_\alpha(z) \;=\; z + \sum_{n=2}^{\infty} A_n z^n \;=\; \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \text{for } \alpha \neq 1/2, \\ \log\frac{1}{1-z}, & \text{for } \alpha = 1/2, \end{cases}$$

whose coefficients are given by

$$A_n = \frac{\Gamma(n+1-2\alpha)}{n!\,\Gamma(2-2\alpha)} \;=\; \frac{1}{n!}\prod_{k=2}^{n}(k-2\alpha).$$

In [1], the Zalcman conjecture was studied in the class $C(-1/2)$. Arguments similar to those we have used earlier allow us to recover and generalize, without much effort, a recent theorem from [53].

Theorem 2.5. *Let $\alpha < 1$, $f \in \overline{\text{co}}(C(\alpha))$, $m, n \geq 2$, and A_n as above. Then*

$$\left|\frac{a_m a_n}{A_m A_n} - \frac{a_{m+n-1}}{A_{m+n-1}}\right| + \frac{|a_m a_n|}{A_m A_n} \leq 1\,.$$

This is equivalent to the following statement:

$$|\lambda a_m a_n - a_{m+n-1}| \leq \max\{A_{m+n-1}, |\lambda A_m A_n - A_{m+n-1}|\} \quad \text{for all} \quad \lambda \in \mathbb{C}\,.$$

Equality holds in both inequalities above for the function given by $f = f_\alpha$ in the case when $|\lambda A_m A_n - A_{m+n-1}| \geq A_{m+n-1}$ and for the function

$$f(z) = \frac{1}{N}\sum_{k=1}^{N} \eta^{-k} f_\alpha\left(\eta^k z\right), \qquad N = m+n-2, \;\; \eta = e^{\frac{2\pi i}{N}}, \tag{2.9}$$

in the case when $|\lambda A_m A_n - A_{m+n-1}| < A_{m+n-1}$.

Proof. In view of formula (11) in Section 0.3 there exists some function in the class $\mathcal{P}$ with coefficients p_n for which the relation

$$a_n = \frac{A_n p_{n-1}}{2}$$

holds. The desired inequalities now follow from Theorem 1.1 and Proposition 1.5.

The function f_α clearly satisfies the equality in the cases indicated. We now compute the coefficients of the function (2.9). We have that

$$\begin{aligned} f(z) &= \frac{1}{N}\sum_{k=1}^{N} \eta^{-k} f_\alpha\left(\eta^k z\right) \\ &= z + \frac{1}{N}\sum_{n=2}^{\infty} A_n \left(\sum_{k=1}^{N} \eta^{(n-1)k}\right) z^n \\ &= z + A_{N+1} z^{N+1} + A_{2N+1} z^{2N+1} + \ldots, \end{aligned}$$

since η is one of the N-th roots of unity.

$\square$

For functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\overline{\text{co}}(S^*(\alpha))$, essentially the same result can be obtained. To state it one simply has to replace A_n for B_n in all inequalities. For example, the first one would read

$$\left|\frac{b_m b_n}{B_m B_n} - \frac{b_{m+n-1}}{B_{m+n-1}}\right| + \frac{|b_m b_n|}{B_m B_n} \leq 1\,.$$

The coefficients B_n are those of the typical example in $\overline{\text{co}}(S^*(\alpha))$:

$$g_\alpha(z) \;=\; z+\sum_{n=2}^{\infty} B_n z^n \;=\; \frac{z}{(1-z)^{2-2\alpha}},$$

and are given by $B_n = nA_n$. The corresponding extremal functions can also be obtained in an identical fashion.

2.3 Asymptotic Zalcman conjecture

Let $f \in S$ and $M_\infty(r,f) = \max_{|z|=r}|f(z)|$. Recall that the Hayman index of f is the number

$$\alpha = \lim_{r\to 1}(1-r)^2 M_\infty(r,f)\,.$$

Even though the Zalcman conjecture continues to be an open problem, we now show that its asymptotic version is true and we give it in a precise quantitative form.

Theorem 2.6. *Let $f(z) = z + a_2z^2 + \dots$ be in S, with Hayman index α, and let $\lambda \in \mathbb{C}$. Then*

$$\lim_{m,n\to\infty}\frac{|\lambda a_m a_n - a_{m+n-1}|}{|\lambda mn - m - n + 1|} = \alpha^2\,. \tag{2.10}$$

Also, if we define $B_{m,n}(\lambda) = \sup_{f\in S}|\lambda a_m a_n - a_{m+n-1}|$, then

$$\lim_{m,n\to\infty}\frac{B_{m,n}(\lambda)}{|\lambda mn - m - n + 1|} = 1.$$

In both limits, we understand that $(m,n)\to(\infty,\infty)$ unconditionally in $\mathbb{N}^2$ (meaning that $m+n\to\infty$).

Proof. Applying the triangle inequality we get

$$\begin{aligned}\frac{|\lambda a_m a_n - a_{m+n-1}|}{|\lambda mn - m - n + 1|} \le{}& \frac{|a_m a_n|}{mn}\,\frac{|\lambda|mn}{|\lambda mn - m - n + 1|}\\ &+\frac{|a_{m+n-1}|}{m+n-1}\,\frac{m+n-1}{|\lambda mn - m - n + 1|},\end{aligned}$$

where the right-hand side converges to α^2 in view of Hayman's regularity theorem. Analogously, we can use the triangle inequality to get a lower bound converging to α^2. Hence (2.10) follows.

The Koebe function clearly shows that $B_{m,n}(\lambda) \geq |\lambda\, mn - m - n + 1|$. Using the customary notation $A_n = \sup_{f\in S} |a_n|$, we have

$$1 \leq \frac{B_{m,n}(\lambda)}{|\lambda\, mn - m - n + 1|} \leq \frac{|\lambda| A_m A_n + A_{m+n-1}}{|\lambda\, mn - m - n + 1|} \to 1\,,$$

when $(m, n) \to (\infty, \infty)$. □

Corollary 2.7. *If $f \in S$ is not a rotation of the Koebe function, then for every $\delta \in (0, 1-\alpha^2)$ there exist m_0 and n_0 in $\mathbb{N}$ (which depend on f) such that*

$$|\lambda a_m a_n - a_{m+n-1}| \leq (1-\delta)|\lambda mn - m - n + 1|$$

for all $m \geq m_0$, $n \geq n_0$.

2.4 Equivalent reformulations and weaker Zalcman conjectures

Some equivalent reformulations of the Zalcman conjecture. For the sake of simplicity, we treat only the original conjecture: $|a_n^2 - a_{2n-1}| \leq (n-1)^2$. We first recall that, if assumed true for all n, it easily implies the Bieberbach conjecture (now de Branges' theorem). Since the proof of this implication for one value of n uses the validity of the conjecture for another n, in order to avoid this discussion in the sequel, we shall simply take for granted the Bieberbach conjecture for odd integers: $|a_{2n-1}| \leq 2n-1$. With this in mind, the Zalcman conjecture can be reformulated in several ways.

Theorem 2.8. *Let $f \in S$ be fixed, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and let $n \geq 2$ be arbitrary. Then the following statements are equivalent:*

(a) The Zalcman conjecture holds: $|a_n^2 - a_{2n-1}| \leq (n-1)^2 = n^2 - (2n-1)$;

(b) $|a_n^2 - t a_{2n-1}| \leq n^2 - t(2n-1)$ for all $t \in [0, 1]$;

(c) $|a_n^2 - a_{2n-1}| + r|a_{2n-1}| \leq (n-1)^2 + r(2n-1)$ for all $r > 0$;

(d) $|a_n^2 - w a_{2n-1}| \leq (n-1)^2 + |w-1|(2n-1)$ for all $w \in \mathbb{C}$.

Proof. We will show that (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b). Of course, other schemes of proof are also possible.

$\boxed{\text{(b)} \Rightarrow \text{(a)}}$. This implication is trivial.

$\boxed{\text{(a)} \Rightarrow \text{(c)}}$. Suppose that (a) holds. In view of the inequality $|a_{2n-1}| \leq 2n-1$, we deduce directly from (a) that

$$|a_n^2 - a_{2n-1}| + r|a_{2n-1}| \leq (n-1)^2 + r(2n-1)$$

for all $r > 0$.

$\boxed{\text{(c)} \Rightarrow \text{(d)}}$. Suppose

$$|a_n^2 - a_{2n-1}| + r|a_{2n-1}| \leq (n-1)^2 + r(2n-1)$$

holds for all $r > 0$ (hence, by taking limits, also for $r = 0$). Let w be arbitrary. If $w = 1$ then (d) follows from the assumption for $r = 0$. For every other value of w there is a positive r such that $|w-1| = r$ and we get

$$\begin{aligned} |a_n^2 - wa_{2n-1}| &= |a_n^2 - a_{2n-1} + (1-w)a_{2n-1}| \\ &\leq |a_n^2 - a_{2n-1}| + r|a_{2n-1}| \\ &\leq (n-1)^2 + r(2n-1) \\ &= (n-1)^2 + |w-1|(2n-1)\,, \end{aligned}$$

and (d) is proved.

$\boxed{\text{(d)} \Rightarrow \text{(b)}}$. This follows readily by taking $w = t \in [0,1]$. $\square$

Several remarks are in order to show that Theorem 2.8 may shed some new light on the problem.

• In view of Theorem 2.8, proving the Zalcman conjecture amounts to proving any of the equivalent statements while disproving it would amount to finding one single example of a function which does not satisfy one of the inequalities (b), (c) or (d) for one single value of t, r or w respectively.

• Statement (b) in the theorem had already been verified for the typically real functions and follows from [13, Theorem 1].

• The fact that Bieberbach's conjecture is true means that (d) holds for $w = 0$. If Zalcman's conjecture were to be true, we would have many more new inequalities such as, for example,

$$|a_n^2 - 2a_{2n-1}| \leq n^2 \,,$$

obtained by taking $w = 2$ in (d).

• We also note that the validity of Bieberbach's conjecture readily implies that (d) is true for any $w = -M$, where M is real and positive; indeed:

$$|a_n^2 + Ma_{2n-1}| \leq n^2 + M(2n-1) = (n-1)^2 + (M+1)(2n-1)\,.$$

However, we do not know whether (d) is true in general for *any other* value of w except for those in $(-\infty, 0]$. So there appears to be a significant gap between Bieberbach and Zalcman.

Three related but weaker conjectures. At this point it seems natural to formulate three closely related conjectures. They could be of interest since they are both weaker than Zalcman's but each of them also implies the Bieberbach conjecture.

In relation to condition (b) of our preceding theorem, for a given value t in $[0,1]$ we will denote by (B_t) the following statement:

$$|a_n^2 - ta_{2n-1}| \leq n^2 - t(2n-1) \qquad (B_t)$$

for all $f \in S$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and all $n \geq 2$. Thus, we can formulate the *first weak* version of the Zalcman conjecture as follows.

Conjecture 1. *There exists $t \in (0,1]$ such that (B_t) holds.*

It is not clear in any obvious way that this statement is true. However, (B_0) is precisely the Bieberbach conjecture and we know it is true. Thus, the set of all $t \in [0,1]$ for which (B_t) holds is non-empty. It is easy to see that this set is closed as the defining condition contains a non-strict inequality. It is also convex; indeed, if (B_s) and (B_t) hold and α, $\beta \in [0,1]$ with $\alpha + \beta = 1$

then clearly

$$\begin{aligned}|a_n^2 - (\alpha s + \beta t)a_{2n-1}| &\le \alpha|a_n^2 - s a_{2n-1}| + \beta|a_n^2 - t a_{2n-1}| \\ &\le \alpha(n^2 - s(2n-1)) + \beta(n^2 - t(2n-1)) \\ &= n^2 - (\alpha s + \beta t)(2n-1)\,,\end{aligned}$$

hence $(B_{\alpha s+\beta t})$ is also true. Thus, it seems natural to consider the quantity $T = \sup\{t \in [0,1] : (B_t)\,\text{is true}\,\}$. With this notation, the Zalcman conjecture claims that $T \ge 1$, while the weak Zalcman conjecture only claims that $T > 0$.

Now consider the situation when condition (c) in Theorem 2.8 holds only for *some* $r > 0$. So for a fixed $r > 0$ we can consider the statement (C_r):

$$|a_n^2 - a_{2n-1}| + r|a_{2n-1}| \le (n-1)^2 + r(2n-1) \tag{C_r}$$

for all $f \in S$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and all $n \ge 2$. This clearly gives rise to the *second weak* version of the Zalcman conjecture.

Conjecture 2. *There exists $r \in [0,1]$ such that (C_r) holds.*

It also makes sense to consider a weaker version of condition (d) in Theorem 2.8. For a fixed r, say $r \in [0,1]$, consider

$$|a_n^2 - w a_{2n-1}| \le (n-1)^2 + |w-1|(2n-1) \quad \text{for all } w \text{ with } |w-1| = r\,, \tag{D_r}$$

for all $f \in S$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and all $n \ge 2$. Thus, we have the *third weak* version of the Zalcman conjecture.

Conjecture 3. *There exists $r \in [0,1]$ such that (D_r) holds.*

The following relationship exists between the conjectures mentioned.

Theorem 2.9. *Assume only a weaker statement than the Bieberbach conjecture, for example, Littlewood's theorem [23, Theorem 2.8]: $|a_n| < en$ for all $n \ge 2$. Under these assumptions we have:*

(a) The Zalcman conjecture implies Conjecture 3.

(b) Conjecture 3 implies Conjecture 2.

(c) Conjecture 2 implies Conjecture 1 (with $t = 1 - r$).

(d) Conjecture 1 implies the Bieberbach conjecture.

(e) All weak conjectures: Conjecture 1, Conjecture 2 and Conjecture 3 are asymptotically true. For example, if f is a function in S with Hayman index α and $t \in [0,1]$ then

$$\lim_{n\to\infty} \frac{|a_n^2 - ta_{2n-1}|}{n^2 - t(2n-1)} = \alpha^2 .$$

Proof. (a) This implication is trivial.

(b) If Conjecture 3 is true, then for the corresponding value of r we have

$$|a_n^2 - wa_{2n-1}| \leq (n-1)^2 + r(2n-1)$$

for all w on the circle $\{w : |w-1| = r\}$. If $a_n - a_{2n-1} \neq 0$ and $a_{2n-1} \neq 0$ we can choose a (unique) w on this circle with $\arg w = \arg(a_n^2 - a_{2n-1}) - \arg a_{2n-1}$ so as to obtain

$$|a_n^2 - wa_{2n-1}| = |a_n^2 - a_{2n-1} + (1-w)a_{2n-1}| = |a_n^2 - a_{2n-1}| + r|a_{2n-1}| ,$$

and (C_r) follows. If any of the values $a_n^2 - a_{2n-1}$, a_{2n-1} is zero, the statement also holds trivially.

(c) Assume that Conjecture 2 is true. For the corresponding $r \in [0,1]$, consider $t = 1 - r \in [0,1]$. Then by the triangle inequality

$$\begin{aligned} |a_n^2 - ta_{2n-1}| &\leq |a_n^2 - a_{2n-1}| + r|a_{2n-1}| \\ &\leq (n-1)^2 + r(2n-1) \\ &\leq n^2 - t(2n-1) , \end{aligned}$$

which proves that Conjecture 1 is true.

(d) To show that Conjecture 1 implies the Bieberbach inequality, we follow the idea of Brown and Tsao from [13]. Begin with a weaker bound for the n-th coefficient, say $|a_n| \leq Cn$, for some $C > 1$, and then improve on it using condition (B_t). As was mentioned, we can start off from Littlewood's theorem and $C = e$. Note that

$$t \leq 1 < \frac{n^2}{2n-1} \quad \text{for} \quad \text{all} \ \ n \geq 2\,.$$

Hence

$$\begin{aligned}
|a_n|^2 &\leq |a_n^2 - ta_{2n-1}| + t|a_{2n-1}| \\
&\leq n^2 - t(2n-1) + Ct(2n-1) \\
&= n^2 + t(C-1)(2n-1) \\
&\leq Cn^2\,.
\end{aligned}$$

Therefore, $|a_n| \leq \sqrt{C}\,n$. Iterating this procedure, we obtain $|a_n| \leq C^{2^{-k}}n$ for all positive integers k, which yields $|a_n| \leq n$.

(e) The proof is quite similar to that of Theorem 2.6 so we omit it. $\square$

Chapter 3

Bombieri's conjecture

3.1 History of the problem and new results

In the class S of normalized univalent functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \qquad z \in \mathbb{D},$$

Bieberbach's conjecture states that $|a_n| \le n$ and that the only extremal functions are the Koebe function

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$$

and its rotations. Louis de Branges succeeded in proving this conjecture in 1984 (see [9] and, also, [42]).

Long before the final solution by de Branges, efforts of many mathematicians culminated in the local proof of Bieberbach's conjecture (*i.e.*, in a neighborhood of the Koebe function) in an article of Bombieri [8]. In the same article, Bombieri conjectured that the numbers

$$\sigma_{mn} = \liminf_{f \to K} \frac{n - \operatorname{Re} a_n}{m - \operatorname{Re} a_m}, \tag{3.1}$$

usually referred to as the *Bombieri numbers*, should coincide with the trigonometric numbers

$$B_{mn} = \min_{t \in \mathbb{R}} \frac{n \sin t - \sin(nt)}{m \sin t - \sin(mt)}$$

for all $m, n \geq 2$. We note that the lower limit in (3.1) refers to functions f in the class S approaching the Koebe function uniformly on compacta.

In [64], Prokhorov and Roth showed that $\sigma_{mn} \leq B_{mn}$. Also, the local maximum property of the Koebe function yields that $\sigma_{mn} \geq 0$. Setting

$$A_n(t) = n - \frac{\sin(nt)}{\sin t}, \quad t \in \mathbb{R},\ n \in \mathbb{N}, \tag{3.2}$$

it is relatively simple to see that $B_{mn} = 0$ when m is even and n is odd, since in that case $A_n(\pi) = 0 < A_m(\pi)$. Hence $\sigma_{mn} = B_{mn} = 0$ and Bombieri's conjecture is correct when m is even and n is odd. Also, the conjecture was verified for functions with real coefficients in [64] and for analytic variations of the Koebe function in [14]. Some related results are given in the recent article [2].

The Bombieri conjecture was first disproved by Greiner and Roth [40] in the case $(m, n) = (3, 2)$. They explicitly computed

$$\sigma_{32} = \frac{e-1}{4e} < \frac{1}{4} = B_{32}.$$

Disproofs for the points $(m, n) = (2, 4), (3, 4)$ and $(4, 2)$ were then furnished by Prokhorov and Vasil'ev [65], who computed (approximately) the corresponding Bombieri numbers.

Recently, Leung [51] developed a variational method which allowed him to show that $\sigma_{m2} < B_{m2}$ for all $m \geq 3$ and that $\sigma_{m3} < B_{m3}$ for all odd $m \geq 5$. He used the *linear* version of Loewner's differential equation

$$\frac{\partial f}{\partial t} = z \frac{\partial f}{\partial z} \frac{1 + \kappa(t) z}{1 - \kappa(t) z}, \tag{3.3}$$

which we have presented in Section 0.2. Recall that in the special case when $\kappa \equiv -1$ the unique solution of (3.3) is the Loewner chain $f(z, t) = e^t K(z)$, whose initial value is the Koebe function. Setting $\kappa(t) = -e^{i\varepsilon\vartheta(t)}$ for $\varepsilon > 0$ and some admissible ϑ and letting $t = 0$, Leung obtained from (3.3) a variation of Koebe's function, given by

$$f(z) = K(z) + \varepsilon v(z) + \varepsilon^2 q(z) + O(\varepsilon^3), \tag{3.4}$$

for some analytic functions v and q which depend only on the choice of ϑ. This way Leung re-derived in a simpler fashion the exact same second variation q as Bombieri, who used the *non-linear* version of Loewner's equation. Thus Bombieri's formula (4.1) in [8] was obtained by Leung as formula (2.17) in [51]. We will present this derivation in Section 3.2.

In terms of the coefficients, formula (3.4) yields

$$a_n = n + \varepsilon v_n + \varepsilon^2 q_n + O(\varepsilon^3).$$

It is an innate property of the method that the coefficients v_n are purely imaginary and q_n are real. Therefore,

$$n - \operatorname{Re} a_n = -\varepsilon^2 q_n + O(\varepsilon^3).$$

Leung's choice of ϑ yields

$$q_n = -\frac{4}{9}(n-1)(2n^2 - 4n + 3). \tag{3.5}$$

In Section 3.3 we will show how to arrive at this q_n via a slightly more direct approach than Leung's. Hence

$$\sigma_{mn} \le \lim_{\varepsilon \to 0^+} \frac{-\varepsilon^2 q_n + O(\varepsilon^3)}{-\varepsilon^2 q_m + O(\varepsilon^3)} = \frac{q_n}{q_m}$$

for all $m, n \ge 2$. Note that

$$\frac{q_n}{q_m} = \frac{(n-1)(2n^2 - 4n + 3)}{(m-1)(2m^2 - 4m + 3)} < \frac{n^3 - n}{m^3 - m}$$

for all $m > n \ge 2$ since

$$\varphi(n) = \frac{2n^2 - 4n + 3}{n(n+1)}$$

increases. Indeed,

$$\varphi'(x) = \frac{3(2x^2 - 2x - 1)}{x^2(x+1)^2} > 0, \qquad \text{for} \quad x > \frac{1+\sqrt{3}}{2} \approx 1,366.$$

Therefore, to disprove Bombieri's conjecture for some $m > n \ge 2$, it suffices to show that

$$B_{mn} = \frac{n^3 - n}{m^3 - m}. \tag{3.6}$$

Leung showed that formula (3.6) holds true for $n = 2$ and for all $m \ge 3$ and, also, for $n = 3$ and for all odd $m \ge 5$. In this chapter we describe the content of our article [29] and prove that identity (3.6) is true in some other cases, including the ones just mentioned. In particular, we will prove the following theorem.

Theorem 3.1. *Let $m > n \geq 2$ be integers such that either*
(a) both m and n are odd, or
(b) both m and n are even, or
(c) m is odd, n is even and $n \leq \frac{m+1}{2}$.
Then (3.6) is true.

We have already observed that one can deduce the following corollary.

Corollary. *Let $m > n \geq 2$ be integers such that either (a), (b) or (c) in Theorem 3.1 holds. Then Bombieri's conjecture for this pair of integers is false.*

Theorem 3.1 will be proved mainly with the use of trigonometry, but also, in the case when the hypothesis (c) holds, we will employ Dieudonné's criterion (mentioned in Section 0.2) for univalent polynomials.

After carefully examining the relevant graphs for $2 \leq n \leq 80$ using the www.desmos.com/calculator software, one is lead to believe that the hypothesis (c) in Theorem 3.1 can be notably weakened in that the point (m, n) has to be below the straight line that joins the points $(7, 6)$ and $(17, 14)$. Thus, the following proposition should be true.

Conjecture. *If $m > n \geq 2$ are integers such that m is odd, n is even and $n < \frac{4m+2}{5}$ then (3.6) is true.*

3.2 Leung's derivation of Bombieri's formula

We will now show how Leung re-derived Bombieri's formula for the second variation of the Koebe function (formula (4.1) in [8]). According to it, if ϕ is a function in $L^2[0, 1]$ then a second variation of the Koebe function is given by $q(z) = Q\big(K(z)\big)$, where

$$Q(w) = -w^2 \int_0^1 \frac{\phi(u)^2}{U} du - 2w^3 \int_0^1 \int_0^u \left(3 + \frac{1}{W}\right) \frac{\phi(u)\phi(v)}{\sqrt{UW}} dv du,$$

$U = 1 + 4uw$ and $W = 1 + 4vw$.

Let $\varepsilon > 0$ and consider the drive function $\kappa(t) = -e^{i\varepsilon\vartheta(t)}$ for some measurable real-valued ϑ on $[0, \infty)$. Expanding with respect to ε we have

$$\kappa = -e^{i\varepsilon\vartheta} = -1 - i\vartheta\varepsilon + \frac{\vartheta^2}{2}\varepsilon^2 + O(\varepsilon^3). \tag{3.7}$$

Consider Loewner's PDE

$$\frac{\partial f}{\partial t}(z,t) \;=\; z\,\frac{\partial f}{\partial z}(z,t)\,\frac{1+\kappa(t)z}{1-\kappa(t)z} \tag{3.8}$$

and let the Loewner chain

$$f(z,t) \;=\; e^t z + a_2(t)z^2 + \dots, \qquad z \in \mathbb{D},\ t \geq 0,$$

be its solution. Since the drive function (3.8) is a variation of -1, the resulting chain must be a variation of the chain whose initial value is the Koebe function, that is,

$$f(z,t) \;=\; e^t K(z) + \varepsilon v(z,t) + \varepsilon^2 q(z,t) + O(\varepsilon^3).$$

To see that the function v is analytic with respect to z we simply apply the $\overline{\partial}$-operator, divide by ε and let $\varepsilon \to 0$. We repeat the process, this time dividing by ε^2, to see that q is analytic. Also, both functions v and q and their first derivatives vanish at the origin. Hence we may write

$$v(z,t) = \sum_{n=2}^{\infty} v_n(t)z^n \qquad \text{and} \qquad q(z,t) = \sum_{n=2}^{\infty} q_n(t)z^n.$$

The upcoming computations will be easier to manipulate in the w-plane, where $w = K(z)$, rather than in the z-plane. Let

$$z = S(w) = \frac{1 + 2w - \sqrt{1+4w}}{2w}, \qquad w \in \mathbb{C}\backslash(-\infty, -1/4],$$

be the inverse of the Koebe function (choosing the branch of the square root so that $\sqrt{1} = 1$) and see that the following elementary but useful identities are true

$$\frac{S}{1-S^2} = \frac{w}{\sqrt{1+4w}} \qquad \text{and} \qquad \frac{S}{(1+S)^2} = \frac{w}{1+4w}. \tag{3.9}$$

We set $F(w,t) = f\big(S(w),t\big)$ and use the chain rule to compute

$$\frac{\partial f}{\partial z} \;=\; \frac{\partial F}{\partial w}\,\frac{\partial w}{\partial z} \;=\; \frac{\partial F}{\partial w}\,\frac{1+z}{(1-z)^3}.$$

A substitution in (3.8) yields

$$\begin{aligned}\frac{\partial F}{\partial t} \;&=\; z\,\frac{\partial F}{\partial w}\,\frac{1+z}{(1-z)^3}\,\frac{1+\kappa z}{1-\kappa z}\\ &=\; w\,\frac{\partial F}{\partial w}\,\frac{1+S}{1-S}\,\frac{1+\kappa S}{1-\kappa S}.\end{aligned} \tag{3.10}$$

In view of (3.7) it is elementary to verify that

$$\frac{1+\kappa S}{1-\kappa S} \;=\; \frac{1-S}{1+S} - \frac{2i\vartheta S}{(1+S)^2}\varepsilon + \frac{\vartheta^2 S(1-S)}{(1+S)^3}\varepsilon^2 + O(\varepsilon^3), \tag{3.11}$$

simply by multiplying both sides by

$$1-\kappa S \;=\; 1+S+i\vartheta S\varepsilon - \frac{\vartheta^2 S}{2}\varepsilon^2 + O(\varepsilon^3)$$

and carrying out the computation. We substitute (3.11) in (3.10) to obtain

$$\frac{\partial F}{\partial t} = w\frac{\partial F}{\partial w}\left(1 - \frac{2i\vartheta S}{1-S^2}\varepsilon + \frac{\vartheta^2 S}{(1+S)^2}\varepsilon^2 + O(\varepsilon^3)\right). \tag{3.12}$$

Setting

$$V(w,t) = v(S(w),t) \qquad \text{and} \qquad Q(w,t) = q(S(w),t)$$

we may write

$$F(w,t) \;=\; f(S(w),t) \;=\; e^t w + \varepsilon V(w,t) + \varepsilon^2 Q(w,t) + O(\varepsilon^3).$$

Equation (3.12) now becomes

$$e^t w+\varepsilon V_t+\varepsilon^2 Q_t = (e^t w+\varepsilon w V_w+\varepsilon^2 w Q_w)\left(1 - \frac{2i\vartheta S}{1-S^2}\varepsilon + \frac{\vartheta^2 S}{(1+S)^2}\varepsilon^2\right)+O(\varepsilon^3),$$

which implies that

$$V_t \;=\; wV_w - 2i\vartheta e^t w\frac{S}{1-S^2}, \tag{3.13}$$

in view of the terms of first order and the fact that

$$Q_t \;=\; wQ_w - 2i\vartheta w V_w\frac{S}{1-S^2} + e^t\vartheta^2 w\frac{S}{(1+S)^2}, \tag{3.14}$$

by comparing the terms of second order.

We start by determining the first variation V. In view of (3.9), equation (3.13) is equivalent to

$$V_t = wV_w - \frac{2i\vartheta e^t w^2}{\sqrt{1+4w}}. \tag{3.15}$$

We set

$$\widetilde{V}(w,t) = V(we^{-t},t) \qquad \text{and} \qquad \widetilde{Q}(w,t) = Q(we^{-t},t)$$

and see that (3.15) becomes

$$\widetilde{V}_t = V_t - we^{-t}V_w = -\frac{2i\vartheta w^2 e^{-t}}{\sqrt{1+4we^{-t}}}.$$

Integrating over $[t,\infty)$ we obtain

$$\lim_{\tau\to\infty} \widetilde{V}(w,\tau) - \widetilde{V}(w,t) = -2iw^2 \int_t^\infty \frac{\vartheta(\tau)e^{-\tau}}{\sqrt{1+4we^{-\tau}}}d\tau.$$

Note that since both $e^{-t}f(\cdot,t)$ and K belong to the class S, we have by the growth theorem that

$$|\varepsilon V(w,t) + \varepsilon^2 Q(w,t) + O(\varepsilon^3)| = |e^{-t}f(z,t) - K(z)| \le \frac{2|z|}{(1-|z|)^2}$$

for every $z = S(w)$ in $\mathbb{D}$. Setting $z(t) = S(we^{-t})$ we see that $z(t)$ converges to the origin as $t \to \infty$. Hence

$$|\varepsilon\widetilde{V}(w,t) + \varepsilon^2\widetilde{Q}(w,t) + O(\varepsilon^3)| \le \frac{2|z(t)|}{(1-|z(t)|)^2} \longrightarrow 0$$

and we can deduce that both $\widetilde{V}(w,t)$ and $\widetilde{Q}(w,t)$ tend to zero as $t \to \infty$. Therefore we have that

$$\widetilde{V}(w,t) = 2iw^2 \int_t^\infty \frac{\vartheta(\tau)e^{-\tau}}{\sqrt{1+4we^{-\tau}}}d\tau$$

and that the first variation, seen in the w-plane, is

$$V(w,0) = \widetilde{V}(w,0) = 2iw^2 \int_0^\infty \frac{\vartheta(\tau)e^{-\tau}}{\sqrt{1+4we^{-\tau}}}d\tau.$$

We now turn to the second variation Q and combine the identities (3.9) with equation (3.14) to obtain

$$Q_t = wQ_w - \frac{2i\vartheta w^2}{\sqrt{1+4w}}V_w + \frac{\vartheta^2 w^2 e^t}{1+4w}.$$

Replacing w by we^{-t} and recalling our definition $\widetilde{Q}(w,t) = Q(we^{-t},t)$ we get

$$\widetilde{Q}_t = Q_t - we^{-t}Q_w = -\frac{2i\vartheta w^2 e^{-2t}}{\sqrt{1+4we^{-t}}}V_w + \frac{\vartheta^2 w^2 e^{-t}}{1+4we^{-t}}.$$

We integrate this over $[t,\infty)$ to obtain

$$\widetilde{Q}(w,t) = 2iw^2 \int_t^\infty \frac{\vartheta(\tau)e^{-\tau}}{\sqrt{1+4we^{-\tau}}} e^{-\tau}V_w(we^{-\tau},\tau)d\tau - w^2 \int_t^\infty \frac{\vartheta^2(\tau)e^{-\tau}}{1+4we^{-\tau}}d\tau.$$

We compute

$$e^{-t}V_w(we^{-t},t) = \widetilde{V}_w(w,t) = 4iw \int_t^\infty \frac{\vartheta(\tau)e^{-\tau}(1+3we^{-\tau})}{(1+4we^{-\tau})^{3/2}}d\tau$$

in order to replace it in the preceding formula. We have

$$\begin{aligned}\widetilde{Q}(w,t) = &- 8w^3 \int_t^\infty \frac{\vartheta(\tau)e^{-\tau}}{\sqrt{1+4we^{-\tau}}} \int_\tau^\infty \frac{\vartheta(s)e^{-s}(1+3we^{-s})}{(1+4we^{-s})^{3/2}}ds\, d\tau \\ &- w^2 \int_t^\infty \frac{\vartheta^2(\tau)e^{-\tau}}{1+4we^{-\tau}}d\tau\end{aligned}$$

and therefore $Q(w,0) = \widetilde{Q}(w,0)$ is our second variation, viewed in the w-plane.

To to see that this is precisely Bombieri's formula we set $u = e^{-\tau}$, $v = e^{-s}$, $\phi(u) = \vartheta(\tau)$ and $\phi(v) = \vartheta(s)$. We have that

$$Q(w,0) = -8w^3 \int_0^1 \int_0^u \frac{\phi(u)\phi(v)(1+3vw)}{\sqrt{1+4uw}(1+4vw)^{3/2}}dvdu - w^2 \int_0^1 \frac{\phi(u)^2}{1+4uw}du.$$

We set $U = 1+4uw$ and $W = 1+4vw$, and note that

$$4(1+3vw) = 1+3W,$$

in order to obtain

$$Q(w,0) = -2w^3 \int_0^1 \int_0^u \left(3+\frac{1}{W}\right)\frac{\phi(u)\phi(v)}{\sqrt{UW}}dvdu - w^2 \int_0^1 \frac{\phi(u)^2}{U}du,$$

which was our objective.

3.3 Coefficients of the second variation of the Koebe function

Here our starting point will be Bombieri's formula (4.1) in [8]. According to it, if ϕ is a function in $L^2[0,1]$ then a second variation of the Koebe function is given by $q(z) = Q\big(K(z)\big)$, where

$$Q(w) = -w^2 \int_0^1 \frac{\phi(u)^2}{U} du - 2w^3 \int_0^1 \int_0^u \left(3 + \frac{1}{W}\right) \frac{\phi(u)\phi(v)}{\sqrt{UW}} dv du, \quad (3.16)$$

$U = 1 + 4uw$ and $W = 1 + 4vw$. Note the following homogeneity property: if we replace ϕ by $c\,\phi$ ($c \in \mathbb{R}$) then instead of Q we obtain $c^2 Q$. In fact, our aim here is to show how a specific choice of ϕ yields

$$q_n = -\frac{1}{9}(n-1)(2n^2 - 4n + 3),$$

which is a scalar multiple of (3.5). We will provide a slightly more direct approach than Leung who, for additional purposes, considered (3.16) with variable $z \in \mathbb{D}$ and integration over the interval $[-1,1]$ in order to use properties of classical orthogonal polynomials.

We rewrite (3.16) as

$$\begin{aligned} Q(w) = & - w^2 \int_0^1 \frac{\phi(u)^2}{1+4uw} du \\ & - 6w^3 \int_0^1 \int_0^u \frac{\phi(u)\phi(v)}{\sqrt{1+4uw}\sqrt{1+4vw}} dv du \\ & - 2w^3 \int_0^1 \int_0^u \frac{\phi(u)\phi(v)}{\sqrt{1+4uw}(1+4vw)^{3/2}} dv du \end{aligned}$$

and denote by I_1, I_2 and I_3 the three integrals in the order appearance, so that

$$Q(w) = -w^2(I_1 + 6wI_2 + 2wI_3).$$

We observe that the integrand in I_2 is symmetric in u and v and therefore its integral over the lower triangle of $[0,1]^2$ (which is I_2) is equal to the integral over the upper triangle. Hence

$$I_2 = \frac{1}{2}\left(\int_0^1 \frac{\phi(u)}{\sqrt{1+4uw}} du\right)^2.$$

To deal with I_3 we note that

$$\frac{2w}{(1+4vw)^{3/2}} = -\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{1+4vw}}\right).$$

An integration by parts now yields

$$\begin{aligned} 2wI_3 = & -\int_0^1 \frac{\phi(u)^2}{1+4uw}du + \phi(0)\int_0^1 \frac{\phi(u)}{\sqrt{1+4uw}}du \\ & + \int_0^1\int_0^u \frac{\phi(u)\phi'(v)}{\sqrt{1+4uw}\sqrt{1+4vw}}dvdu. \end{aligned}$$

In total, we have

$$\begin{aligned} Q(w) = & -w^2\phi(0)\int_0^1 \frac{\phi(u)}{\sqrt{1+4uw}}du - 3w^3\left(\int_0^1 \frac{\phi(u)}{\sqrt{1+4uw}}du\right)^2 \\ & - w^2\int_0^1\int_0^u \frac{\phi(u)\phi'(v)}{\sqrt{1+4uw}\sqrt{1+4vw}}dvdu. \end{aligned} \tag{3.17}$$

We now choose $\phi(u) = 1-u$. It is helpful to compute

$$\int_0^u \frac{dv}{\sqrt{1+4vw}} = \frac{\sqrt{1+4uw}-1}{2w}$$

and (integrating by parts):

$$\int_0^1 \frac{u\,du}{\sqrt{1+4uw}} = \frac{\sqrt{1+4w}}{2w} - \frac{(1+4w)^{3/2}-1}{12w^2}.$$

Then we can compute the integrals in (3.17). They are

$$\int_0^1 \frac{\phi(u)}{\sqrt{1+4uw}}du = \frac{(1+4w)^{3/2}-6w-1}{12w^2}$$

and

$$\int_0^1\int_0^u \frac{\phi(u)\phi'(v)}{\sqrt{1+4uw}\sqrt{1+4vw}}dvdu = \frac{(1+4w)^{3/2}-6w^2-6w-1}{24w^3}.$$

We substitute these in (3.17) and after elementary but cumbersome calculations we obtain

$$Q(w) = \frac{1+4w}{6}\left(\sqrt{1+4w}-1-2w\right).$$

Setting $w = K(z) = \frac{z}{(1-z)^2}$ we get

$$q(z) = Q\big(K(z)\big) = -\frac{z^2(1+z)^2}{3(1-z)^4}.$$

Finally, we compute the n-th coefficient of q with the aid of the standard formula

$$\frac{1}{(1-z)^4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} z^n.$$

3.4 Auxiliary lemmas

We first prove a simple lemma for $A_n(t) = n - \frac{\sin(nt)}{\sin t}$, which we have defined in (3.2).

Lemma 3.2. *For all $t \in \mathbb{R}$ and $n \geq 2$, we have*

$$A_n(t) \geq 0 \qquad \textit{and} \qquad A_n(2\pi - t) = A_n(t).$$

Also, A_n vanishes only for $t = 2\ell\pi, \ell \in \mathbb{Z}$, when n is even and only for $t = \ell\pi, \ell \in \mathbb{Z}$, when n is odd.

Proof. The symmetry is fairly obvious. Due to it we may restrict our attention to the interval $[0, \pi]$.

Using L'Hospital's rule we find that

$$A_{2k}(0) = A_{2k+1}(0) = A_{2k+1}(\pi) = 0, \qquad A_{2k}(\pi) = 4k$$

for any $k \geq 1$. Now, for $t \in (0, \pi)$, $A_n(t) > 0$ is equivalent to

$$\varphi(t) := n \sin t - \sin(nt) > 0,$$

whose derivative is

$$\varphi'(t) = n(\cos t - \cos(nt)).$$

If t_0 is a critical point of φ then $\sin t_0 = \pm \sin(nt_0)$. Hence

$$\varphi(t_0) = (n \mp 1) \sin t_0 > 0$$

and the proof is complete. $\square$

We remark that in this lemma, for odd n we could use the connection with the Dirichlet kernel

$$D_n(x) = \frac{\sin(n+1/2)x}{\sin x/2} = 1 + 2\sum_{j=1}^{n} \cos(jx),$$

which is $A_{2k+1}(t) = 2k+1-D_k(2t)$ (see [25, §8.4], for example). For even n we would simply have to adjust the proof of the above expansion in cosines, where the trick with telescoping sums works equally well.

Lemma 3.3. *For all integers $n \geq 2$ and for all $t \in (0, \pi)$ it holds that*

$$\frac{A_n(t)}{n^3-n} \geq \frac{A_{n+2}(t)}{(n+2)^3-(n+2)}. \tag{3.18}$$

Proof. We set $N = n+1 \geq 3$ and see that (3.18) is equivalent to

$$N(N+1)(N+2)A_{N-1}(t) \;\geq\; N(N-1)(N-2)A_{N+1}(t),$$

which, in turn, is equivalent to

$$4(N^2-1) - (N+1)(N+2)\frac{\sin(N-1)t}{\sin t} + (N-1)(N-2)\frac{\sin(N+1)t}{\sin t} \geq 0.$$

Multiplying by $\frac{1}{2}\sin t$, expanding the sines of the sums and setting

$$\Phi(t) \;=\; 2(N^2-1)\sin t - 3N\sin(Nt)\cos t + (N^2+2)\cos(Nt)\sin t, \tag{3.19}$$

we see that the above is equivalent to $\Phi(t) \geq 0$. We note that

$$\Phi\left(\frac{\pi}{2}\right) = 2N^2 - 2 + (N^2+2)\cos\left(\frac{N\pi}{2}\right) \geq N^2 - 4 > 0$$

since shortly we will need to consider $t \neq \frac{\pi}{2}$. We compute

$$\begin{aligned}\frac{\Phi'(t)}{N^2-1} \;&=\; 2\cos t - 2\cos(Nt)\cos t - N\sin(Nt)\sin t \\ &=\; 2\sin\left(\frac{Nt}{2}\right)\left(2\sin\left(\frac{Nt}{2}\right)\cos t - N\cos\left(\frac{Nt}{2}\right)\sin t\right).\end{aligned} \tag{3.20}$$

Hence, one set of the roots of Φ' comes from $\sin\left(\frac{Nt}{2}\right) = 0$. The solutions of this equation satisfy $Nt_k = 2k\pi, k \in \mathbb{Z}$, and it is easy to check that

$$\Phi(t_k) = 3N^2 \sin t_k > 0.$$

The rest of the roots of Φ' comes from

$$\tan\left(\frac{Nt}{2}\right) = \frac{N}{2}\tan t, \tag{3.21}$$

if we momentarily consider that $\cos\left(\frac{Nt}{2}\right) \neq 0$. We return to (3.19) and compute

$$\Phi(t) \;=\; (N^2-4)\sin t + 2\cos^2\left(\frac{Nt}{2}\right)\sin t\left(N^2+2-3N\frac{\tan\left(\frac{Nt}{2}\right)}{\tan t}\right).$$

Hence, if t^* satisfies (3.21) then

$$\Phi(t^*) \;=\; (N^2-4)\sin t^*\left(1-\cos^2\left(\frac{Nt^*}{2}\right)\right) \geq 0,$$

which was our goal. Therefore, it is only left to consider the case when $\cos\left(\frac{Nt}{2}\right) = 0$ for some critical point of Φ. But this would give $Nt = (2k+1)\pi, k \in \mathbb{Z}$, and a substitution in (3.20) yields

$$\frac{\Phi'(t)}{N^2-1} = 4\cos t,$$

which vanishes only at $t = \frac{\pi}{2}$, a point we have previously considered. □

3.5 Proof of the main theorem

We now proceed with the proof of our main theorem.

Proof of Theorem 3.1. We set

$$\varphi_{mn}(t) := \frac{n\sin t - \sin(nt)}{m\sin t - \sin(mt)} = \frac{A_n(t)}{A_m(t)}, \quad t \in [0, 2\pi],$$

whose minimum is the number B_{mn}. In view of the symmetry of A_n (stated in Lemma 3.2) we may restrict our attention to t in $[0, \pi]$.

Suppose first that either the hypothesis (a) or (b) holds, that is, m and n are simultaneously odd or even. Note that

$$\varphi_{mn}(0) = \varphi_{mn}(\pi) = \frac{n^3-n}{m^3-m} \qquad \text{for odd } \; m, n$$

and that

$$\varphi_{mn}(0) = \frac{n^3 - n}{m^3 - m} < \frac{n}{m} = \varphi_{mn}(\pi) \qquad \text{for even } m, n.$$

Hence, our goal is to show that

$$\frac{A_n(t)}{A_m(t)} \geq \frac{n^3 - n}{m^3 - m} \qquad \text{for } t \in (0, \pi).$$

But this follows directly from Lemma 3.3 after a finite number of iterations

$$\frac{A_n(t)}{n^3 - n} \geq \frac{A_{n+2}(t)}{(n+2)^3 - (n+2)} \geq \frac{A_{n+4}(t)}{(n+4)^3 - (n+4)} \geq \cdots \geq \frac{A_m(t)}{m^3 - m}.$$

Suppose now that the hypothesis (c) holds, that is, m is odd, n is even and $n \leq \frac{m+1}{2}$. Note that

$$\varphi_{mn}(0) = \frac{n^3 - n}{m^3 - m} < +\infty = \varphi_{mn}(\pi).$$

Once again, in view of Lemma 3.3 it suffices to prove that

$$\frac{A_n(t)}{A_{m_0}(t)} \geq \frac{n^3 - n}{m_0^3 - m_0} \qquad \text{for } t \in (0, \pi),$$

where $m_0 = 2n - 1$. This is equivalent to

$$4(2n-1)A_n(t) \geq (n+1)A_{2n-1}(t),$$

which, in turn, is the same as

$$1 - \frac{4}{3n-1}\frac{\sin(nt)}{\sin t} + \frac{n+1}{(2n-1)(3n-1)}\frac{\sin\big((2n-1)t\big)}{\sin t} \geq 0. \tag{3.22}$$

It would clearly suffice to prove that

$$1 - \frac{4}{3n-1}\frac{\sin(nt)}{\sin t}z^{n-1} + \frac{n+1}{(2n-1)(3n-1)}\frac{\sin\big((2n-1)t\big)}{\sin t}z^{2n-2} \neq 0 \tag{3.23}$$

for all $z \in \mathbb{D}$, since this would imply that for $z = x \in [0, 1)$ the function in (3.23) is positive and (3.22) would follow after letting $x \to 1^-$. In view of Dieudonné's criterion (Lemma 0.5 in Section 0.2), (3.23) is equivalent to the statement that the function

$$f(z) = z - \frac{4}{3n-1}z^n + \frac{n+1}{(2n-1)(3n-1)}z^{2n-1}$$

belongs to the class S. We will actually prove more: we will show that f is starlike, which means that f is univalent and that for every $w \in f(\mathbb{D})$ the line segment $[0, w]$ lies entirely in $f(\mathbb{D})$.

First, we see that the roots of

$$\frac{f(z)}{z} = 1 - \frac{4}{3n-1} z^{n-1} + \frac{n+1}{(2n-1)(3n-1)} z^{2n-2}$$

satisfy

$$z^{n-1} = \frac{2(2n-1) \pm i(n-1)\sqrt{3(2n-1)}}{n+1},$$

and therefore

$$|z|^{2n-2} = \frac{(2n-1)(3n^2+2n-1)}{(n+1)^2} > 1.$$

This shows that the function

$$p(z) = \frac{zf'(z)}{f(z)}$$

is analytic in $\overline{\mathbb{D}}$ and so in order to apply the well-known criterion for starlikeness it suffices to show that

$$\operatorname{Re} p(z) \geq 0 \quad \text{for} \quad |z| = 1. \tag{3.24}$$

We compute

$$\frac{p(z)}{2n-1} = \frac{(n+1)z^{2n-2} - 4nz^{n-1} + 3n - 1}{(n+1)z^{2n-2} - 4(2n-1)z^{n-1} + (2n-1)(3n-1)}$$

and let $z^{n-1} = e^{i\theta}$, $\theta \in \mathbb{R}$. We then have

$$\begin{aligned} \frac{p(z)}{2n-1} &= \frac{(n+1)e^{i\theta} - 4n + (3n-1)e^{-i\theta}}{(n+1)e^{i\theta} - 4(2n-1) + (2n-1)(3n-1)e^{-i\theta}} \\ &= \frac{2n(\cos\theta - 1) - (n-1)i\sin\theta}{(3n^2-2n+1)\cos\theta - 2(2n-1) - 3n(n-1)i\sin\theta}. \end{aligned}$$

Multiplying by the complex conjugate of the denominator we see that (3.24) is equivalent to

$$\begin{aligned} 0 &\leq 2n(\cos\theta - 1)\big[(3n^2-2n+1)\cos\theta - 2(2n-1)\big] + 3n(n-1)^2\sin^2\theta \\ &= n(n+1)(3n-1)(\cos\theta - 1)^2, \end{aligned}$$

which is true. The proof is complete. □

Chapter 4

Harmonic Bloch-type mappings

4.1 Overview

Since the mid-1980s and especially after the seminal article of Clunie and Sheil-Small [18], harmonic mappings -which up to that point were studied mainly by differential geometers due to their role in parametrizing minimal surfaces- have started to attract the attention of complex analysts. The basic observation was that many classical results for conformal mappings have clear analogues for univalent harmonic mappings. Thereafter, great effort was put into extending function theory of analytic functions to harmonic mappings. In this spirit, our goal in this chapter is to define and study a harmonic analogue of the analytic Bloch space $\mathcal{B}$. This is the content of our article [30].

Metric definition of Bloch functions. A relevant reference in this direction is the work of F. Colonna [19], whose point of departure was the *metric* characterization of analytic Bloch functions, namely, $f \in \mathcal{B}$ if and only if f is Lipschitz between $\mathbb{D}$ endowed with the hyperbolic metric and $\mathbb{C}$ endowed with the euclidean metric. It was proved in [19] that for a harmonic mapping $f = h + \overline{g}$, this Lipschitz condition is equivalent to both h and g belonging to $\mathcal{B}$. Also, it was shown that all bounded harmonic mappings satisfy the above Lipschitz condition.

Geometric definition of Bloch functions. The *radius of univalence*, or *schlicht radius*, $d_f(z)$ of a harmonic mapping $f = h + \overline{g}$ is defined as the radius of the largest disk which is the injective image of some subdomain of $\mathbb{D}$ and is centered at $f(z)$. We set $d_f(z) = 0$ if no such disk exists. A generalization of the *geometric* definition of Bloch functions would be to ask that f satisfy $\sup_{z\in\mathbb{D}} d_f(z) < \infty$. It is an interesting problem to characterize analytically this condition. We will prove in Lemma 4.4 that if $f = h + \overline{g}$ is univalent and normalized then

$$\frac{1}{16}(1-|z|^2)(|h'(z)|-|g'(z)|) \;\leq\; d_f(z) \;\leq\; \frac{\pi}{2}(1-|z|^2)|h'(z)|, \qquad z \in \mathbb{D}.$$

However, examples can be furnished showing that no two of the above three quantities are comparable. Therefore, an analytic characterization of the geometric definition for harmonic mappings is, as far as we know, yet to be found.

A new class of harmonic Bloch-type mappings. Our starting point will be the *analytic* definition (20) from Section 0.5. Noting that the Jacobian of an analytic function φ is given by $J_\varphi = |\varphi'|^2$, it seems natural to introduce the following definition.

Definition 1. Let $f = h+\overline{g}$ be harmonic in $\mathbb{D}$. We say that f is a *Bloch-type mapping* if

$$\beta(f) \;=\; \sup_{z\in\mathbb{D}}(1-|z|^2)\sqrt{|J_f(z)|} < \infty.$$

We denote this class of mappings by $\mathcal{B}_H$.

Indeed, we will see in Section 4.2 that this definition gives rise to a class rather than a linear space. However, $\mathcal{B}_H$ contains the Bloch space defined by Colonna in [19]. We will prove that $\mathcal{B}_H$ is both affine and linearly invariant. In Section 4.3 we will find a connection between $\mathcal{B}_H$ and univalent harmonic mappings that resembles Pommerenke's theorem 0.7 from Section 0.5. We will also study the radius of univalence in $\mathcal{B}_H$. In Section 4.4 we will give growth and coefficients estimates for sense-preserving mappings in $\mathcal{B}_H$.

4.2 Basic properties of harmonic Bloch-type mappings

Our first task will be to prove the affine and linear invariance of $\mathcal{B}_H$. As before, we will denote by $\varphi_\alpha\,(\alpha \in \mathbb{D})$ the disk automorphism given by $\varphi_\alpha(z) = (\alpha + z)/(1 + \overline{\alpha} z)$, $z \in \mathbb{D}$.

Proposition 4.1. *If $f \in \mathcal{B}_H$ then*

(i) $af + b\overline{f} \in \mathcal{B}_H$ for any $a, b \in \mathbb{C}$ (affine invariance).

(ii) $f \circ \varphi_\alpha \in \mathcal{B}_H$ for any $\alpha \in \mathbb{D}$ (linear invariance).

Proof. Let $f = h + \overline{g}$. To prove (i) we write

$$F = af + b\,\overline{f} = ah + bg + \overline{\overline{a}g + \overline{b}h}$$

and compute

$$J_F = |ah' + bg'|^2 - |\overline{a}g' + \overline{b}h'|^2 = (|a|^2 - |b|^2)\, J_f.$$

The assertion now easily follows.

In order to prove claim (ii), we write $F = f \circ \varphi_\alpha = H + \overline{G}$ and compute

$$H'(z) = \frac{h'\big(\varphi_\alpha(z)\big)(1 - |\alpha|^2)}{(1 + \overline{\alpha} z)^2}, \qquad G'(z) = \frac{g'\big(\varphi_\alpha(z)\big)(1 - |\alpha|^2)}{(1 + \overline{\alpha} z)^2}.$$

Hence

$$\begin{aligned}(1 - |z|^2)\sqrt{|J_F(z)|} &= \frac{(1 - |z|^2)(1 - |\alpha|^2)}{|1 + \overline{\alpha} z|^2}\sqrt{|J_f\big(\varphi_\alpha(z)\big)|} \\ &= \big(1 - |\varphi_\alpha(z)|^2\big)\sqrt{|J_f\big(\varphi_\alpha(z)\big)|}.\end{aligned}$$

Taking the supremum over $z \in \mathbb{D}$ we get that $\beta(F) = \beta(f)$. $\square$

In what follows, Example 1 shows that $\mathcal{B}_H$ is not a linear space. It also shows that mappings in $\mathcal{B}_H$ may grow arbitrarily fast. Hence, in order to get growth and coefficient estimates in Section 4.4 we shall restrict our attention to the sense-preserving mappings in $\mathcal{B}_H$.

Example 1. Consider an analytic function h for which $h'(z) = (1-z)^{-p}$, for some $p > 2$. Set $f = h + \overline{h} = 2\,\mathrm{Re}\,\{h\}$ and see that, since $J_f \equiv 0$, f belongs to $\mathcal{B}_H$. Obviously, the identity $\mathrm{id}(z) = z$ belongs to $\mathcal{B}_H$, but we will see that $f + \mathrm{id}$ does not. Indeed,

$$J_{f+\mathrm{id}} = |h' + 1|^2 - |h'|^2 = 1 + 2\mathrm{Re}\,\{h'\}$$

and therefore, for $0 < x < 1$ we have

$$(1-x^2)^2|J_{f+\mathrm{id}}(x)| = (1+x)^2 \frac{2 + (1-x)^p}{(1-x)^{p-2}} \longrightarrow \infty$$

as $x \to 1^-$.

Example 2 shows that the harmonic Bloch space considered by Colonna in [19] is strictly contained in $\mathcal{B}_H$. Recall that in [19] the definition of a Bloch mapping $f = h + \overline{g}$ is equivalent to both h and g belonging to $\mathcal{B}$.

Example 2. Let $f = h + \overline{g}$ be given by $h(z) = \frac{2}{\sqrt{1-z}}$ and $\omega(z) = g'(z)/h'(z) = z$. Then $f \in \mathcal{B}_H$ since $h'(z) = (1-z)^{-3/2}$ and

$$(1-|z|^2)\sqrt{J_f(z)} = \left(\frac{1-|z|^2}{|1-z|}\right)^{3/2} \leq 2\sqrt{2}.$$

Note that $h \notin \mathcal{B}$ since, for $0 < x < 1$, we have

$$(1-x^2)|h'(x)| = \frac{1+x}{\sqrt{1-x}} \longrightarrow \infty$$

as $x \to 1^-$. Therefore f is not a Bloch mapping for [19].

4.3 Connections to univalent harmonic mappings

Analogues of Pommerenke's theorem. The well-known Theorem 0.7 of Pommerenke [61] states that an analytic function f is Bloch if and only if there exists a constant $\alpha \in \mathbb{C}$ and a function $g \in S$ such that $f = \alpha \log g' + f(0)$. The following theorems show a similar connection between harmonic univalent mappings and the class $\mathcal{B}_H$.

Theorem 4.2. *Let $F = H + \overline{G}$ be univalent and sense-preserving in $\mathbb{D}$. Let $h = \log(H')$ and consider any $\omega : \mathbb{D} \to \mathbb{D}$ analytic. Then $f = h + \overline{g}$, having dilatation $\omega_f = \omega$, belongs to $\mathcal{B}_H$.*

Proof. Let $\alpha \in \mathbb{D}$ and compose F with a disk automorphism φ_α to obtain

$$T(z) = \frac{F\left(\frac{\alpha+z}{1+\overline{\alpha}z}\right) - F(\alpha)}{(1-|\alpha|^2)H'(\alpha)}.$$

It can easily be seen that $T \in S_H$ and that the second coefficient of the analytic part of T is given by

$$a_2(\alpha) = (1-|\alpha|^2)\frac{H''(\alpha)}{2H'(\alpha)} - \overline{\alpha}.$$

We turn to $f = h + \overline{g}$ and compute

$$\begin{aligned}
(1-|\alpha|^2)\sqrt{J_f(\alpha)} &\leq (1-|\alpha|^2)|h'(\alpha)| \\
&= (1-|\alpha|^2)\left|\frac{H''(\alpha)}{H'(\alpha)}\right| \\
&= 2|a_2(\alpha) + \overline{\alpha}| \\
&< 2\left(48.4 + \frac{1}{2}\right) + 2 \\
&= 99.8,
\end{aligned}$$

in view of (22). The proof is complete. □

We have another theorem for the opposite direction, but first we need the following definition. For any analytic $\omega : \mathbb{D} \to \mathbb{D}$ we define its *hyperbolic derivative* by

$$\omega^*(z) = \frac{\omega'(z)(1-|z|^2)}{1-|\omega(z)|^2}.$$

We set $\|\omega\|_h = sup_{z\in\mathbb{D}}|\omega^*(z)|$ for its *hyperbolic norm* (see [4, §5]).

Although the following theorem is formulated for the class $\mathcal{B}_H$, it should be noted that it is in fact about the smaller set of harmonic mappings considered by Colonna [19]. The set of mappings $f = h + \overline{g} \in \mathcal{B}_H$ for which g belongs to $\mathcal{B}$ is precisely Colonna's space of harmonic Bloch mappings.

Theorem 4.3. *Let $f = h + \overline{g} \in \mathcal{B}_H$ be sense-preserving and suppose that $g \in \mathcal{B}$. Let $0 < \varepsilon < 1$. Set*

$$H(z) = \int_0^z \exp\Big(\frac{\varepsilon}{c}\, h(\zeta)\Big)\, d\zeta,$$

where $c = \sqrt{\beta(g)^2 + \beta(f)^2}$, and consider any analytic $\omega : \mathbb{D} \to \mathbb{D}$ satisfying $\|\omega\|_h \le (1-\varepsilon)/2$. Then $F = H + \overline{G}$, having dilatation $\omega_F = \omega$, is univalent.

Proof. We apply Theorem 0.8 to the mapping F. Since $f \in \mathcal{B}_H$ and $g \in \mathcal{B}$, we have that

$$(1 - |z|^2)^2 |h'(z)|^2 \;\le\; \beta(f)^2 + (1 - |z|^2)^2 |g'(z)|^2 \;\le\; c^2.$$

Hence

$$\left|\frac{H''(z)}{H'(z)}\right| \;=\; \frac{\varepsilon}{c}|h'(z)| \;\le\; \frac{\varepsilon}{1 - |z|^2}.$$

Also, the definition of the hyperbolic norm and our hypothesis lead to

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \;\le\; \frac{\|\omega\|_h}{1 - |z|^2} \;\le\; \frac{1 - \varepsilon}{2(1 - |z|^2)}.$$

We may now compute

$$|zP_F(z)| + \frac{|z\omega_F'(z)|}{1 - |\omega_F(z)|^2} \;\le\; \left|\frac{H''(z)}{H'(z)}\right| + \frac{2|\omega'(z)|}{1 - |\omega(z)|^2} \;\le\; \frac{1}{1 - |z|^2}$$

and conclude that F is univalent by Theorem 0.8. $\square$

Radius of univalence. As we have already mentioned, the *radius of univalence* $d_f(z)$ of a harmonic mapping $f = h + \overline{g}$ at a point $z \in \mathbb{D}$ is defined as the radius of the largest disk which is the injective image of some subdomain of $\mathbb{D}$ and is centered at $f(z)$. If there is no such disk then we set $d_f(z) = 0$. The existence of a universal lower bound for $\sup_{z\in\mathbb{D}} d_f(z)$ in a given class of functions is commonly refered to as a *Bloch theorem* for this class. It was shown in [17] that openness (*i.e.*, the property of mapping open sets to open sets) and the normalization $g'(0) = 1 - h'(0) = 0$ are sufficient conditions for a Bloch theorem to hold. Moreover, an example was furnished showing that the normalization alone is not a sufficient condition.

Since here we will be concerned only with univalent mappings, the radius of univalence coincides with the distance between $f(z)$ and the boundary of $f(\mathbb{D})$. The following lemma provides us with some estimates.

Lemma 4.4. *If $f \in S_H$ then*

$$\frac{1}{16}(1-|z|^2)(|h'(z)|-|g'(z)|) \;\le\; d_f(z) \;\le\; \frac{\pi}{2}(1-|z|^2)|h'(z)|$$

for all $z \in \mathbb{D}$.

Proof. Let $\alpha \in \mathbb{D}$ and compose with a disk automorphism φ_α to obtain

$$F(z) = \frac{f\left(\frac{\alpha+z}{1+\overline{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)h'(\alpha)} = H(z) + \overline{G(z)}.$$

Since $F \in S_H$, the covering theorem (23) and Hall's result (24) from Section 0.6 imply that the radius $d_F(0)$ of the largest disk centered at the origin and contained in the image of F satisfies

$$\frac{1-|B_1|}{16} \;\le\; d_F(0) \;\le\; \frac{\pi}{2}.$$

We compute

$$d_F(0) = \frac{d_f(\alpha)}{(1-|\alpha|^2)|h'(\alpha)|}$$

and $B_1 = g'(\alpha)/\overline{h'(\alpha)}$, the first coefficient of G. The inequality follows upon substitution. $\square$

Theorem 4.5. *Let $f \in S_H$.*

(i) If $f \in \mathcal{B}_H$ then $d_f(z) = O\left(\frac{1}{\sqrt{1-|z|}}\right), |z| \to 1^-$.

(ii) If $d_f(z) = O\left(\sqrt{1-|z|}\right), |z| \to 1^-$, then $f \in \mathcal{B}_H$.

If, in addition, f is a quasiconformal mapping then $f \in \mathcal{B}_H$ if and only if $\sup_{z\in\mathbb{D}} d_f(z) < \infty$.

We shall need the following standard lemma. See [34], page 3.

Lemma 4.6. *If $\omega : \mathbb{D} \to \mathbb{D}$ is analytic then*

$$|\omega(z)| \le \frac{|\omega(0)|+|z|}{1+|\omega(0)||z|}.$$

Proof of Theorem 4.5. Note that $f \in \mathcal{B}_H$ is equivalent to

$$(1-|z|^2)|h'(z)|\sqrt{1-|\omega(z)|^2} \leq \beta(f), \quad z \in \mathbb{D}.$$

Also note that $\omega(0) = b_1$. An application of Lemma 4.4 and Lemma 4.6 yields

$$d_f(z) \;\leq\; \frac{\pi}{2}(1-|z|^2)|h'(z)| \;\leq\; \frac{\pi}{2}\frac{\beta(f)}{\sqrt{1-|\omega(z)|^2}} \;\leq\; \frac{\pi}{2}\sqrt{\frac{1+|b_1|}{1-|b_1|}}\frac{\beta(f)}{\sqrt{1-|z|}},$$

so that claim (i) is proved. For assertion (ii) we use again lemmas 4.4 and 4.6 to get

$$(1-|z|^2)\sqrt{J_f(z)} \;\leq\; 16\, d_f(z)\sqrt{\frac{1+|\omega(z)|}{1-|\omega(z)|}} \;\leq\; 16\sqrt{2}\sqrt{\frac{1+|b_1|}{1-|b_1|}}\frac{d_f(z)}{\sqrt{1-|z|}},$$

hence $f \in \mathcal{B}_H$.

Suppose now that f is quasiconformal and see that its dilatation $\omega = g'/h' : \mathbb{D} \to \mathbb{D}$ satisfies

$$\|\omega\|_\infty \;=\; \sup_{z\in\mathbb{D}} |\omega(z)| \;<\; 1.$$

Arguing as before but using only Lemma 4.4 we get

$$d_f(z) \;\leq\; \frac{\pi}{2}\frac{\beta(f)}{\sqrt{1-\|\omega\|_\infty}}$$

and in the opposite direction

$$(1-|z|^2)\sqrt{J_f(z)} \;\leq\; 16\, d_f(z)\sqrt{\frac{1+\|\omega\|_\infty}{1-\|\omega\|_\infty}}.$$

The proof is complete. □

4.4 Growth and coefficients estimates

For a harmonic sense-preserving mapping $f = h + \overline{g}$ with dilatation $\omega = g'/h' : \mathbb{D} \to \mathbb{D}$, we write

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \omega(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Of course $c_0 = b_1/a_1$. We will also make use of the standard notation

$$M_\infty(r, f) = \max_{|z|=r} |f(z)|.$$

We now present some growth and coefficients estimates for the class $\mathcal{B}_H$. Note, however, that these bounds are not uniform throughout $\mathcal{B}_H$, but rather, to each of its subclasses having prescribed $|c_0|$.

Theorem 4.7. *If $f = h + \overline{g} \in \mathcal{B}_H$ is sense-preserving then*

$$\max\{|h(z) - a_0|, |g(z)|\} \leq \beta(f)\sqrt{\frac{1+|c_0|}{1-|c_0|}}\frac{r}{\sqrt{1-r^2}}, \quad |z| = r.$$

This estimate is sharp in order of magnitude.

Proof. Let $|z| = r < 1$ and write

$$h(z) - a_0 = \int_0^z h'(\zeta)d\zeta = z\int_0^1 h'(tz)dt.$$

We have

$$|h(z) - a_0| \leq r\int_0^1 |h'(tz)|dt \leq r\int_0^1 \frac{\beta(f)}{(1-r^2t^2)\sqrt{1-|\omega(tz)|^2}}dt,$$

since $f \in \mathcal{B}_H$. We use Lemma 4.6 to get

$$|h(z) - a_0| \leq \beta(f)\sqrt{\frac{1+|c_0|}{1-|c_0|}}\, r\int_0^1 \frac{dt}{(1-r^2t^2)^{3/2}}.$$

We compute the elementary integral

$$\int_0^1 \frac{dt}{(1-r^2t^2)^{3/2}} = \frac{1}{\sqrt{1-r^2}}$$

and thus complete the proof of the desired inequality for the function h.

We easily get the same bound for g by computing

$$|g(z)| \leq r\int_0^1 |g'(tz)|dt$$

and using the fact that $|g'| \leq |h'|$.

We now prove the sharpness of the order of magnitude. When $c_0 = 0$, both inequalities (for functions h and g) are optimal in view of Example 2. Our considerations here will contain this as a special case. We take $f = h + \overline{g}$, for which $h'(z) = (1-z)^{-3/2}$, as in Example 2, but here we take the dilatation to be a self-map of $\mathbb{D}$ whose image is a horodisk centered at some $t \in [0,1)$, that is, $\omega(z) = (g'/h')(z) = t + (1-t)z$. We see that $f \in \mathcal{B}_H$ since

$$\begin{aligned}(1-|z|^2)\sqrt{J_f(z)} &= \frac{1-|z|^2}{|1-z|^{3/2}}\sqrt{1-|\omega(z)|^2} \\ &= \frac{1-|z|^2}{|1-z|}\sqrt{\frac{1-|z|^2-2t\mathrm{Re}\left(\overline{z}(1-z)\right)-t^2|1-z|^2}{|1-z|}} \\ &\le 2\sqrt{2}\sqrt{1+t}.\end{aligned}$$

The sharpness of the inequality for h is now obvious since $h(z) = \frac{2}{\sqrt{1-z}}$ in our example.

For the function g of this example we compute

$$g'(z) = \frac{1}{(1-z)^{-3/2}} - \frac{1-t}{\sqrt{1-z}}.$$

Integrating we get

$$g(z) = \frac{2}{\sqrt{1-z}} + 2(1-t)\sqrt{1-z},$$

hence, for every $\varepsilon > 0$ we have that

$$(1-x)^{1/2-\varepsilon}|g(x)| \longrightarrow \infty,$$

when $x \to 1^-$. The proof is complete. □

Theorem 4.8. *If $f = h + \overline{g} \in \mathcal{B}_H$ is sense-preserving then*

$$|a_1| \le \frac{\beta(f)}{\sqrt{1-|c_0|^2}}$$

and

$$\max\{|a_n|, |b_n|\} \le \beta(f)\left(\frac{e}{3}\right)^{3/2}\sqrt{\frac{1+|c_0|}{1-|c_0|}}\sqrt{n+2}, \quad n \ge 2.$$

Proof. For the first inequality we put $z = 0$ in the definition of $\mathcal{B}_H$ and get

$$\sqrt{|a_1|^2 - |b_1|^2} \le \beta(f).$$

Let $n \geq 2$. By Cauchy's formula we have that

$$|a_n| = \frac{|h^{(n)}(0)|}{n!} = \left| \frac{1}{n\, 2\pi i} \int_{|\zeta|=r} \frac{h'(\zeta)}{\zeta^n} d\zeta \right| \leq \frac{M_\infty(r, h')}{n\, r^{n-1}}$$

for any $r \in (0,1)$. Similarly, and also due to the fact that f is sense-preserving, we have that

$$|b_n| \leq \frac{M_\infty(r, g')}{n\, r^{n-1}} \leq \frac{M_\infty(r, h')}{n\, r^{n-1}}.$$

The definition of $\mathcal{B}_H$ implies that

$$\max\{|a_n|, |b_n|\} \leq \frac{\beta(f)}{n\, r^{n-1}(1-r^2)\sqrt{1 - M_\infty^2(r, \omega)}}.$$

Using Lemma 4.6 we get that

$$\begin{aligned} \max\{|a_n|, |b_n|\} &\leq \frac{\beta(f)(1 + |c_0| r)}{n\, r^{n-1}(1-r^2)^{3/2}\sqrt{1 - |c_0|^2}} \\ &\leq \frac{\beta(f)}{n} \sqrt{\frac{1+|c_0|}{1-|c_0|}} \frac{1}{r^{n-1}(1-r^2)^{3/2}}. \end{aligned}$$

This inequality is true for all r in $(0,1)$. Therefore, in order to minimize the expression on the right-hand side we see that $r^{n-1}(1-r^2)^{3/2}$ is maximized for $r = \sqrt{\frac{n-1}{n+2}}$. Making this choice we get

$$\begin{aligned} \max\{|a_n|, |b_n|\} \leq & \frac{\beta(f)}{n} \sqrt{\frac{1+|c_0|}{1-|c_0|}} \left(\frac{n+2}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{n+2}{3}\right)^{3/2} \\ = & \frac{\beta(f)}{3\sqrt{3}} \sqrt{\frac{1+|c_0|}{1-|c_0|}} \varphi(n)\sqrt{n+2}, \end{aligned}$$

where

$$\varphi(x) = \left[\left(1 + \frac{3}{x-1}\right)^{\frac{x-1}{3}} \right]^{3/2} \left(1 + \frac{2}{x}\right).$$

Note that $\varphi(x) \to e^{3/2}$ when $x \to +\infty$. We will now show that φ increases to its limit. First note that $\varphi(x) > 0$ for $x \geq 2$. We compute

$$\log \varphi(x) = \frac{x-1}{2} \log\left(\frac{x+2}{x-1}\right) + \log\left(\frac{x+2}{x}\right).$$

Differentiating we get

$$\psi(x) := \frac{\varphi'(x)}{\varphi(x)} = \frac{1}{2}\log\left(\frac{x+2}{x-1}\right) - \frac{3x+4}{2x(x+2)}.$$

One more differentiation yields

$$\psi'(x) = -\frac{x^2+8}{2x^2(x+2)^2(x-1)},$$

which for $x \geq 2$ obviously satisfies $\psi' < 0$. Therefore ψ decreases, so that

$$\psi(x) > \lim_{x\to\infty} \psi(x) = 0,$$

hence $\varphi' > 0$ and the proof is complete. $\square$

Conclusions

This work could provide a basis for further research. For example, the content of Chapter 1 could perhaps find some applications to other extremal problems, possibly to the Krzyż conjecture.

The Zalcman conjecture is far from being solved so further analysis could lead to a large body of work, not only for special classes such as close-to-convex functions but also for other small values of n, for example, $n = 6$, and also to some new results on the related conjectures mentioned at the end of Chapter 2.

It should still be possible to provide further counterexamples to the Bombieri conjecture, at least for some of the remaining pairs of integers (m, n). This would complement our findings in Chapter 3 but may require further time and effort. In some cases a different or deeper analysis may be required.

Finally, regarding Chapter 4, several questions remain regarding the different harmonic Bloch classes and comparisons between them, especially in relation to the one defined by Colonna [19] and to the geometric description of harmonic Bloch mappings via the radius of univalence.

The content of this thesis is the basis of the following articles and preprints:

- I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* **435** (2016), 369-379. Chapter 1 of this thesis.

- I. Efraimidis, D. Vukotić, Applications of Livingston-type inequalities to the generalized Zalcman functional, preprint (arXiv:1611.00682v3). An earlier (unpublished) version of this manuscript is: On the generalized Zalcman functional for some classes of univalent functions (arXiv:1403.5240v1). Chapter 2 of this thesis.

- I. Efraimidis, On the failure of Bombieri's conjecture for univalent functions, preprint (arXiv: 1612.07242v2). Chapter 3 of this thesis.

- I. Efraimidis, J. Gaona, R. Hernández, O. Venegas, On harmonic Bloch-type mappings, *Complex Var. Elliptic Equ.*, published online, doi: 10.1080/17476933.2016.1265951. Chapter 4 of this thesis.

Conclusiones

Este trabajo podría servir como base para investigacion en el futuro. Por ejemplo, el contenido del Capítulo 1 quizás podría ser utilizado para encontrar algunas aplicaciones a otros problemas extremales, posiblemente a la conjetura de Krzyż.

La conjetura de Zalcman está lejos de ser resuelta, por lo que un análisis posterior podría dirigir a un gran trabajo, no sólo para clases especiales como la de funciones cercanas a convexas, sino también para otros valores pequeños de n, por ejemplo, $n = 6$, y también a algunos nuevos resultados sobre las conjeturas mencionadas al final del Capítulo 2.

Todavía debería ser posible aportar más contraejemplos a la conjetura de Bombieri, al menos para algunos de los restantes pares de enteros (m, n). Esto complementaría los hallazgos del Capítulo 3, pero puede requerir más tiempo y esfuerzo. En algunos casos puede ser necesario un análisis diferente o más profundo.

Finalmente, con respecto al Capítulo 4, quedan varias preguntas sobre las diferentes clases de Bloch armónicas y comparaciones entre ellas, especialmente en relación con la definida por Colonna [19] y a la descripción geométrica de las aplicaciones armónicas de Bloch via el radio de univalencia.

El contenido de esta tesis es la base de los siguientes artículos y prepublicaciones:

- I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* **435** (2016), 369-379. Capítulo 1 de esta tesis.

- I. Efraimidis, D. Vukotić, Applications of Livingston-type inequalities to the generalized Zalcman functional, preprint (arXiv:1611.00682v3). Una versión anterior (no publicada) de este manuscrito es: On the generalized Zalcman functional for some classes of univalent functions (arXiv:1403.5240v1). Capítulo 2 de esta tesis.

- I. Efraimidis, On the failure of Bombieri's conjecture for univalent functions, preprint (arXiv: 1612.07242v2). Capítulo 3 de esta tesis.

- I. Efraimidis, J. Gaona, R. Hernández, O. Venegas, On harmonic Bloch-type mappings, *Complex Var. Elliptic Equ.*, published online, doi: 10.1080/17476933.2016.1265951. Capítulo 4 de esta tesis.

Bibliography

[1] Y. Abu-Muhanna, L. Li, S. Ponnusamy, Extremal problems on the class of convex functions of order $-1/2$, *Arch. Math. (Basel)* **103** (2014), no. 6, 461-471.

[2] D. Aharonov, D. Bshouty, A problem of Bombieri on univalent functions, *Comput. Methods Funct. Theory* **16** (2016), no. 4, 677-688.

[3] J.M. Anderson, J. Clunie, Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12-37.

[4] A. Beardon, D. Minda, The hyperbolic metric and geometric function theory, *Quasiconformal mappings and their applications*, 9-56, Narosa, New Delhi, 2007.

[5] J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, *J. Reine Angew. Math.* **255** (1972), 23-43.

[6] J. Becker, Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math.* **354** (1984), 74-94.

[7] H. Bohr, A theorem concerning power series, *Proc. London Math. Soc. (Ser. 2)* **13**, no. 1190 (1914), 1-5.

[8] E. Bombieri, On the local maximum property of the Koebe function, *Invent. Math.* **4** (1967), 26-67.

[9] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), no. 1-2, 137-152.

[10] L. Brickman, T.H. MacGregor, D.R. Wilken, Convex hulls of some classical families of univalent functions, *Trans. Amer. Math. Soc.* **156** (1971), 91-107.

[11] L. Brickman, D.J. Hallenbeck, T.H. MacGregor, D.R. Wilken, Convex hulls and extreme points of families of starlike and convex mappings, *Trans. Amer. Math. Soc.* **185** (1973), 413-428 (1974).

[12] J.E. Brown, Successive coefficients of functions with positive real part, *Int. J. Math. Anal.* **4** (2010), no. 49-52, 2491-2499.

[13] J.E. Brown, A. Tsao, On the Zalcman conjecture for starlike and typically real functions, *Math. Z.* **191** (1986), 467-474.

[14] D. Bshouty, W. Hengartner, A variation of the Koebe mapping in a dense subset of S, *Canad. J. Math.* **39** (1987), no. 1, 54-73.

[15] J.T.P. Campschroer, *Inverse coefficients and symmetrization of univalent functions*, Doctoral Thesis, Katholieke Universiteit Nijmegen, 1984.

[16] C. Carathéodory, Über den variabilitätsbereich der Fourier'schen konstanten von positiven harmonischen funktionen, *Rend. Circ. Mat. Palermo.* **32** (1911), 193-217.

[17] H. Chen, P.M. Gauthier, W. Hengartner, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* **128** (2000), no. 11, 3231-3240.

[18] J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A.I* **9** (1984), 3-25.

[19] F. Colonna, The Bloch constant of bounded harmonic mappings, *Indiana Univ. Math. J.* **38** (1989), no. 4, 829-840.

[20] N. Danikas, Some Banach spaces of analytic functions, *Function spaces and complex analysis*, Joensuu, 1997, Univ. Joensuu Dept. Math. Rep. Ser. **2**, pp. 9-35.

[21] P. Delsarte, Y. Genin, A simple proof of Livingston's inequality for Carathéodory functions, *Proc. Amer. Math. Soc.* **107** (1989), no. 4, 1017-1020.

[22] J. Dieudonné, Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe, *Ann. Sci. École Norm. Sup.* **48** (1931), 247-358.

[23] P.L. Duren, *Univalent Functions*, Springer-Verlag, Berlin-New York, 1983.

[24] P.L. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, 2004.

[25] P.L. Duren, *Invitation to Classical Analysis*, American Mathematical Society, Providence, RI, 2012.

[26] P.L. Duren, M. Schiffer, The theory of the second variation in extremum problems for univalent functions, *J. Analyse Math.* **10** (1962/63) 193-252.

[27] P.L. Duren, A.P. Schuster, *Bergman Spaces*, MSM **100**, AMS, Providence, RI, 2004.

[28] I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* **435** (2016), 369-379.

[29] I. Efraimidis, On the failure of Bombieri's conjecture for univalent functions, submitted (arXiv: 1612.07242v2).

[30] I. Efraimidis, J. Gaona, R. Hernández, O. Venegas, On harmonic Bloch-type mappings, *Complex Var. Elliptic Equ.*, published online, doi: 10.1080/17476933.2016.1265951.

[31] I. Efraimidis, D. Vukotić, On the generalized Zalcman functional for some classes of univalent functions (arXiv:1403.5240v1).

[32] I. Efraimidis, D. Vukotić, Applications of Livingston-type inequalities to the generalized Zalcman functional, submitted (arXiv:1611.00682v3).

[33] M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen. *J. London Math. Soc.* **8** (1933), 85-89.

[34] J.B. Garnett, *Bounded Analytic Functions*, revised first edition, Springer, New York, 2007.

[35] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, RI, 1969.

[36] A.W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.* **8** (1957), 598-601.

[37] A.W. Goodman, *Univalent Functions*, Vol. I, Mariner Publ. Co., Tampa, 1983.

[38] I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Monographs and Textbooks in Pure and Applied Mathematics, 255. Marcel Dekker, Inc., New York, 2003.

[39] U. Grenander, G. Szegö, *Toeplitz Forms and Their Applications*, second edition, Chelsea, New York, 1984.

[40] R. Greiner, O. Roth, On support points of univalent functions and a disproof of a conjecture of Bombieri, *Proc. Amer. Math. Soc.* **129** (2001), no. 12, 3657-3664.

[41] R.R. Hall, A class of isoperimetric inequalities, *J. Analyse Math.* **45** (1985), 169-180.

[42] W.K. Hayman, *Multivalent Functions*, Second edition, Cambridge University Press, Cambridge, 1994.

[43] P. Henrici, *Applied and Computational Complex Analysis, Vol. 1*, Wiley-Interscience, New York, 1974.

[44] R. Hernández, M.J. Martín, Pre-Schwarzian and Schwarzian derivatives of harmonic mappings, *J. Geom. Anal.* **25** (2015), no. 1, 64-91.

[45] F. Holland, The extreme points of a class of functions with positive real part, *Math. Ann.* **202** (1973), 85-87.

[46] W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.* **1** (1952), 169-185.

[47] F.R. Keogh, E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.* **20** (1969), 8-12.

[48] R.A. Kortram, The extreme points of a class of functions with positive real part, *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997), no. 4, 449-459.

[49] S.L. Krushkal, Univalent functions and holomorphic motions, *J. Analyse Math.* **66** (1995), 253-275.

[50] S.L. Krushkal, Proof of the Zalcman conjecture for initial coefficients, *Georgian Math. J.* **17** (2010), no. 4, 663-681. (Erratum in *Georgian Math. J.* **19** (2012), no. 4, 777.)

[51] Y.J. Leung, On the Bombieri numbers for the class S, *J.Anal.*, published online, doi: 10.1007/s41478-016-0017-2.

[52] L. Li, S. Ponnusamy, On the generalized Zalcman functional $\lambda a_n^2 - a_{2n-1}$ in the close-to-convex family, *Proc. Amer. Math. Soc.* **145** (2017), no. 2, 833-846.

[53] L. Li, S. Ponnusamy, J. Qiao, Generalized Zalcman conjecture for convex functions of order α, *Acta Math. Hungar.* **150** (2016), no. 1, 234-246.

[54] R.J. Libera, E.J. Złotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* **85** (1982), no. 2, 225-230.

[55] A.E. Livingston, The coefficients of multivalent close-to-convex functions, *Proc. Amer. Math. Soc.* **21** (1969), 545-552.

[56] A.E. Livingston, A coefficient inequality for functions of positive real part with an application to multivalent functions, *Pacific J. Math.* **120** (1985), no. 1, 139-151.

[57] W. Ma, The Zalcman conjecture for close-to-convex functions, *Proc. Amer. Math. Soc.* **104** (1988), 741-744.

[58] W. Ma, Generalized Zalcman conjecture for starlike and typically real functions, *J. Math. Anal. Appl.* **234** (1999), 328-339.

[59] T.H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.* **104** (1962), 532-537.

[60] Z. Nehari, *Conformal Mapping*, Dover Publications Inc., New York, 1975. Reprinting of the 1952 edition.

[61] Ch. Pommerenke, On Bloch functions, *J. London Math. Soc. (2)* **2** (1970), 689-695.

[62] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[63] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.

[64] D. Prokhorov, O. Roth, On the local extremum property of the Koebe function, *Math. Proc. Cambridge Philos. Soc.* **136** (2004), no. 2, 301-312.

[65] D. Prokhorov, A. Vasil'ev, Optimal control in Bombieri's and Tammi's conjectures, *Georgian Math. J.* **12** (2005), no. 4, 743-761.

[66] M.S. Robertson, On the theory of univalent functions, *Ann. of Math. (2)* **37** (1936), no. 2, 374-408.

[67] G. Schober, *Univalent Functions - Selected Topics*, Lecture Notes in Mathematics, Vol. 478. Springer-Verlag, Berlin-New York, 1975.

[68] J. Seidel, L. Walsh, On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence, *Trans. Amer. Math. Soc.*, **52** (1942), 128-216.

[69] T.J. Suffridge, Some special classes of conformal mappings, *Handbook of Complex Analysis: Geometric Function Theory, Vol.2*, 309-338, Elsevier, Amsterdam, 2005.

[70] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[71] T. Umezawa, Analytic functions convex in one direction, *J. Math. Soc. Japan* **4** (1952), 194-202.